ON THE BOUNDS OF CONVEXITY AND STARLIKENESS FOR FUNCTIONS UNIVALENT AND REGULAR IN A DISK

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Abstract

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MATHEMATICS

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ON THE BOUNDS OF CONVEXITY AND STARLIKENESS FOR FUNCTIONS UNIVALENT AND REGULAR IN A DISK

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Below we give theorems generalizing the well-known propositions of Nevanlinna ([1]) and Grunsky ([2]) that any holomorphic univalent function $z = f(w)$, $f(0) = 0$, $f'(0) = 1$, in the disk $W: \ |w| < 1$ (the totality of such functions is called the class S) maps the disk with center at $w = 0$ and radius not exceeding $R_k = 2 - \sqrt{3}$ onto a convex domain, and the disk with the same center and radius not exceeding $R_s = \operatorname{th} \frac{\pi}{4}$ onto a domain starlike with respect to $z = 0$, as well as theorems refining and supplementing results obtained by G. M. Goluzin ([3,4]) for bounds of generalized starlikeness.

Theorem 1. The disk with center at the point $\omega \in W$ and radius $\rho \leq R_k$,

$$R_k = 2 - \sqrt{3 + |\omega|^2},$$

is mapped by any function $z = f(w)$ of the class S onto a convex domain. The number R_k cannot be increased without additional restrictions on $f(w)$.

The proof is based on the known fact: the range of values of the expression

$$I = \frac{(w - \omega) f''(w)}{f'(w)},$$

where w and ω are fixed points of W, and $f(w) \in S$, is the interior of the disk of radius

$$\frac{4|w - \omega|}{1 - |w|^2}$$

with center at the point

$$\frac{2(w - \omega)|w|^2}{w(1 - |w|^2)}.$$

In the proof of the remaining theorems we use the following lemma, obtained by the variational method.

Lemma. The range of values of the expression

$$I = \ln \frac{(w-\omega) f'(w)}{f(w) - f(\zeta)}$$

($w, \omega, \zeta \in W$ and fixed) in the class S is the interior of the disk of radius

$$\ln \frac{|1-\bar{\zeta} w| + |w-\zeta|}{|1-\bar{\zeta} w| - |w-\zeta|},$$

equal to the non-Euclidean distance between the points ζ and w, with center at the point

$$\ln \frac{(1-|\zeta|^2)(w-\omega)}{(1-w\bar{\zeta})(w-\zeta)}.$$

Theorem 2. Every non-Euclidean disk with non-Euclidean center at the point $\zeta \in W$, whose non-Euclidean radius is not greater than $\pi/2$, is mapped by any function of the class S onto a domain starlike with respect to $f(\zeta)$. The estimate is sharp.

Following G. M. Goluzin, we shall call a domain of the form $D_n(a)$ $(n = 1, 2, ...)$ any closed domain all of whose points can be joined to the point a by a polygonal line lying entirely in it and consisting of no more than n straight-line segments.

Theorem 3. *Every non-Euclidean disk with non-Euclidean center at the point $\zeta \in W$, whose non-Euclidean radius is not greater than $n\pi/2$, is mapped by any function $f(w) \in S$ onto a domain of the form $D_n(f(\zeta))$. The estimate is sharp.*

In particular, every disk of radius not greater than

$$R_{ns} = \operatorname{th} \frac{n\pi}{4}$$

with center at the point $w = 0$ is mapped by every function of the class S onto a domain of the form $D_n(0)$.

Let us note that the proof of Theorems 2 and 3 can be carried out without using the lemma.

Theorem 4. *Every disk which contains the point $w = 0$ on its boundary and has radius not greater than $1/4$ is mapped by any function of the class S onto a domain starlike with respect to $z = 0$. The indicated bound is sharp.*

Theorem 5. *Every disk with center at the point $ae^{i\alpha}$, $a < 1/4$, $0 \leq \alpha \leq 2\pi$, containing the point $w = 0$ in its interior, is mapped by any function of the* *class S onto a domain starlike with respect to $z = 0$, if the radius of the disk r satisfies the inequality*

$$\ln \frac{1 - a + r(\sin x - \cos x)}{1 - a - r(\sin x + \cos x)} \leq \operatorname{arc\,tg} \frac{r + a\cos x}{a \sin x}, \tag{1}$$

where x, $0 < x < \pi$, is determined from the relation

$$\sqrt{r^2 - 2a\cos x + a^2} = \frac{r \sin x}{1 - a - r\cos x}. \tag{2}$$

The estimate is sharp.

Denote by $R(a, 0)$ the upper bound of the radii of disks satisfying (1), (2). It can be shown that the value $R(a, 0)$ is not smaller than the value of the larger root of the equation

$$\ln \frac{1 + a + \rho}{1 - a - \rho} = \arccos \frac{a}{\rho}.$$

Theorem 6. *Every disk with center at the point $w = 0$ is mapped by any function of the class S onto a domain starlike with respect to the image of the point ζ belonging to it, $|\zeta| = a$, if the radius of the disk r satisfies the inequality*

$$\ln \frac{\sqrt{1 - 2ar\cos x + a^2r^2} + \sqrt{r^2 - 2ar\cos x + a^2}}{\sqrt{1 - 2ar\cos x + a^2r^2} - \sqrt{r^2 - 2ar\cos x + a^2}} \leq$$

$$\leq \operatorname{arc\,tg} \frac{r(1 + a^2) - a(1 + r^2)\cos x}{a(1 - r^2)\sin x},$$

where x, $0 < x < \pi$, is determined from the relation

$$\frac{a(1 - r^2)\left[r(1 + a^2)\cos x - a(1 + r^2)\right]}{[r(1 + a^2) - a(1 + r^2)\cos x]^2 + a^2(1 - r^2)^2\sin^2 x} = \frac{2ar \sin x}{\sqrt{(1 - 2ar\cos x + a^2r^2)(r^2 - 2ar\cos x + a^2)}}.$$

The estimate is sharp.

For the upper bound of the radii $R(0, a)$ of the disks named in the theorem, one can indicate a sufficiently precise value from below. It is equal to the larger root of the equation

$$\ln \frac{(1 + a)(1 + \rho)}{(1 - a)(1 - \rho)} = \arccos \frac{a(1 - \rho^2)}{\rho(1 - a^2)}.$$

Theorem 7. Every disk with center at the point $\zeta = ae^{i\alpha} \in W$, whose radius satisfies the inequality

$$\ln \frac{(1-a)(\sin x + \cos x) - r}{(1-a)(\sin x - \cos x) + r} \leq \operatorname{arc\,tg} \frac{1 - a^2 - ar\cos x}{ar \sin x},$$

where $x,\ 0 < x < \pi$, is determined from the relation

$$\sqrt{(1-a^2)^2 - 2ar(1-a^2)\cos x + a^2r^2} = \frac{r(1-a)\sin x}{(1-a)\cos x - r},$$

is mapped by every function of the class S onto a domain star-shaped with respect to the image of the center. The estimate is sharp.

The root of the equation

$$\ln \frac{(1-a)(1+a+\rho)}{(1+a)(1-a-\rho)} = \arccos \frac{a\rho}{1-a^2}$$

gives a sufficiently accurate lower value for the upper bound $R(a, a)$ of the radii of these disks.

Tomsk State University
named after V. V. Kuibyshev

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CITED LITERATURE

[1] R. Nevanlinna, *Oversckt av Finska Vet. Soc. Forh.* (A), **62**, 1 (1919-1920).
[2] Grunsky, *Jahresber. deutsch. Math. Ver.*, **43**, 140 (1934).
[3] G. M. Goluzin, *Matem. sborn.*, **42**, No. 2, 169 (1935).
[4] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, 1952.

Note: Figure translations are in progress. See original paper for figures.