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Z. S. Agranovich and V. A. Marchenko

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Abstract

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Z. S. Agranovich and V. A. Marchenko

RECOVERY OF THE POTENTIAL FROM THE SCATTERING MATRIX FOR A SYSTEM OF DIFFERENTIAL EQUATIONS

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The present note is devoted to the inverse problem of scattering theory for a system of differential equations of the form

$$y''_\alpha + \lambda^2 y_\alpha = \sum_{\beta=1}^{n} v_{\alpha\beta}(x) y_\beta, \qquad 0 < x < \infty \quad (\alpha = 1, 2, \dots, n). \tag{A}$$

This problem was considered in the work (^{1}), where it is reduced to finding the spectral matrix with the subsequent use of the equation of I. M. Gel' fand and B. M. Levitan (^{2}). We give a direct solution of this problem, using the method set forth for a single equation in the work (^{3}). We note that this problem for a single equation was considered by other methods by M. G. Krein (^{4}).

1. Thus, consider the system (A), equivalent to the matrix equation

$$Y'' + \lambda^2 Y = V(x) Y, \tag{1}$$

and suppose that the potential matrix $V(x) = \|v_{\alpha\beta}(x)\|_1^n$ is Hermitian and satisfies the condition

$$\int_0^\infty t\,|V(t)|\,dt < \infty. \tag{2}$$

The absolute value $|A|$ of any matrix $A = \|a_{\alpha\beta}\|_1^n$ is defined by the formula

$$|A| = \max_\alpha \sum_{\beta=1}^{n} |a_{\alpha\beta}|.$$

From condition (2) it obviously follows that, for every $x > 0$, there exists the integral

$$\sigma(x) = \int_0^\infty |V(t)|\,dt. \tag{3}$$

Theorem 1. *Equation (1) has, for every* λ *from the half-plane* $\operatorname{Im}\lambda \le 0$, *a solution* $E(x,\lambda)$ *representable in the form*

$$E(x,\lambda) = e^{-i\lambda x} I + \int_x^\infty K(x,t) e^{-i\lambda t}\,dt, \tag{4}$$

where I *is the identity matrix, and the matrix* $K(x,t)$ *satisfies the inequality*

$$|K(x,t)| \le C\sigma\left(\frac{x+t}{2}\right) \quad (C = \text{const}).$$

Moreover,

$$2K(x,x) = \int_x^\infty V(t)\,dt, \qquad 0 < x < \infty.$$

An analogous theorem for a single equation was first obtained by B. Ya. Levin (^{5}) under somewhat stronger assumptions on the potential.

2. Consider the boundary-value problem defined by system (A) and the condition

$$y_\alpha(0,\lambda) = 0, \qquad \alpha = 1, 2, \dots, n, \tag{B}$$

and denote by $G(x,\lambda)$ the matrix solution of equation (1) satisfying the initial conditions $G(0,\lambda) = 0$, $G'(0,\lambda) = I$. The boundary-value problem (A)–(B) has a finite number p of eigenvalues of the form λ_k^2, $\lambda_k = -i\mu_k$, $\mu_k > 0$, and the corresponding normalized eigenvector-functions are the columns of the matrices

$$U(x,\lambda_k) = E(x,\lambda_k) \cdot M_k, \tag{5}$$

where M_k is a Hermitian matrix whose rank is equal to the multiplicity of the eigenvalue λ_k^2. We shall call M_k the **normalizing matrix**. For real $\lambda \ne 0$ the equality

$$U(x,\lambda) = E(x,-\lambda) - E(x,\lambda)\cdot S(-\lambda) = 2i\lambda G(x,\lambda)\cdot[E^*(0,-\lambda)]^{-1}, \tag{6}$$

holds, where $S(\lambda)$ is a unitary matrix, called the **scattering matrix**.

Theorem 2. *The matrix* $K(x,y)$ *(see Theorem 1) satisfies, for* $0 \le x < y < \infty$, *the equation*

$$F(x+y)+K(x,y)+\int_x^\infty K(x,t)F(t+y)\,dt=0, \tag{7}$$

where $F(u)$ is determined in terms of $S(\lambda)$ and M_k by the formula:

$$F(u)=\sum_{k=1}^{p}M_k^2e^{-\mu_k u}+\frac{1}{2\pi}\int_{-\infty}^{\infty}[I-S(\lambda)]e^{i\lambda u}\,d\lambda.$$

Equation (7) can be obtained by contour integration from equality (6). From equation (7) one easily derives Parseval' s equality for the system of matrix functions (5), (6), equivalent to the following expansion of the δ-function:

$$\delta(x-y)\cdot I=\sum_{k=1}^{p}U(x,\lambda_k)U^*(y,\lambda_k)+\frac{1}{2\pi}\int_0^\infty U(x,\lambda)U^*(y,\lambda)\,d\lambda, \tag{8}$$

Conversely, from formula (8) one can obtain equation (7), using the idea of orthogonalization.

Theorem 3. *The scattering matrix $S(\lambda)$ has the following properties:*

I$_s$. *The matrix $I-S(\lambda)$ is the Fourier transform of a Hermitian matrix $F_1(u)$, so that*

$$F_1(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[I-S(\lambda)]e^{i\lambda u}\,d\lambda. \tag{9}$$

Moreover, the elements of the matrix $F_1(u)$, summable on the half-line $(0,\infty)$, and on the half-line $(-\infty,0)$, are representable as the sum of two functions, one of which is summable, while the other is bounded and square-summable.

II$_s$. *The equation*

$$-x(t)+\int_{-\infty}^{0}x(\xi)F_1(t+\xi)\,d\xi=0,\qquad -\infty<t\le 0, \tag{10}$$

has no nonzero solutions.

III$_S$. For every $u>0$ there exists $F'(u)$ and

$$\int_0^\infty u\,|F'(u)|\,du<\infty.$$

The normalizing matrices M_k and the eigenvalues λ_k^2 ($\lambda_k=-i\mu_k$) are such that:

IV. The equation

$$x(t) + \int_0^\infty x(\xi) F(t+\xi)\, d\xi = 0, \qquad 0 \le t < \infty,$$

where

$$F(u) = \sum_{k=1}^{p} M_k^2 e^{-\mu_k u} + F_1(u), \tag{11}$$

has no nonzero solutions.

V. The number of linearly independent solutions of the equation

$$x(t) + \int_0^\infty x(\xi) F_1(t+\xi)\, d\xi = 0, \qquad 0 \le t < \infty, \tag{12}$$

is equal to the sum of the ranks of the matrices M_k.

Let us note that for the matrix $F(u)$ the inequality

$$|4F'(2u) - V(u)| \le C\sigma^2(u), \tag{13}$$

holds, where C is a constant and $\sigma(x)$ is defined by formula (3), from which property III_S of Theorem 3 follows.

3. We pass to the inverse problem. Suppose Hermitian matrices M_k, numbers $\mu_k > 0$ $(k = 1, 2, \ldots, p)$, and a unitary matrix $S(\lambda)$ possessing property I_S of Theorem 3 are prescribed arbitrarily. From these data, according to formulas (9), (11), construct the matrix $F(u)$, and from it equation (7) with unknown matrix $K(x, y)$. Then the following two theorems hold.

Theorem 4. Equation (7), for all $x > 0$, has a unique solution $K(x, y)$, and the system of matrix functions $U(x, \lambda)$ and $U(x, \lambda_k)$, $\lambda_k = -i\mu_k$, constructed by means of this solution $K(x, y)$ according to formulas (4), (5), (6), gives the expansion of the δ-function (8), equivalent to Parseval' s equality.

If, in addition, the matrix $S(\lambda)$ also possesses property III_S, then the matrix $K(x, x)$ is differentiable for every $x > 0$, and the matrices $U(x, \lambda)$, $U(x, \lambda_k)$ are solutions of equation $(1)^*$ with potential matrix

$$V(x) = -2\frac{d}{dx} K(x, x),$$

which for every $\varepsilon > 0$ satisfies the inequality:

$$|V(x) - 4F'(2x)| \le C(\varepsilon)\tau^2(2x), \tag{14}$$

where $C(\varepsilon)$ is a constant, and

$$\tau(x) = \int_x^\infty |F'(u)|\,du.$$

If, moreover, condition IV of Theorem 3 is also fulfilled, then inequality (14) remains valid also for $\varepsilon = 0$ ($C(0) < \infty$), and then $V(x)$ satisfies condition (2).

Theorem 5. If $F(u) = F_1(u)$, i.e. all $M_k = 0$, then equation (7), constructed from $F(u)$, is solved for each $x > 0$ by the method of successive approximations.

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[*] If one allows the elements of the potential matrix to be generalized functions, then this result is valid under the single condition I_S.

Let us note, incidentally, one more case in which equation (7) is solved elementarily. Namely, if the elements of $S(\lambda)$ are rational functions, then the kernel of equation (7), $F(t+y)$, is degenerate.

It follows from inequalities (13) and (14) that the behavior of the potential matrix $V(x)$ at infinity and near zero is essentially the same as that of the matrix $F'(2x)$.

4. Let us observe that even in the case when all the conditions of Theorem 4 are fulfilled (i.e., I_S, III_S, and IV), the matrices $U(x,\lambda)$ and $U(x,\lambda_k)$, generally speaking, will not satisfy the boundary condition, i.e., may fail to vanish at zero. In order that the matrices $U(x,\lambda)$ and $U(x,\lambda_k)$ vanish at zero, some additional conditions are necessary.

Theorem 6. *If arbitrarily prescribed $S(\lambda)$, M_k, and μ_k possess properties I_S, IV, and V, then the matrices $U(x,\lambda_k)$ satisfy the boundary condition $U(0,\lambda_k) = 0$, $k = 1, 2, \ldots, p$. If, in addition, condition II_S is also fulfilled, then for the matrix $U(x,\lambda)$ we have $U(0,\lambda) = 0$, $0 < \lambda < \infty$.*

From Theorems 3, 4, and 6 there follows the following principal result:

Conditions I_S, II_S, III_S, IV, and V are necessary and sufficient in order that the prescribed unitary matrix $S(\lambda)$, Hermitian matrices M_k, and numbers $-\mu_k^2$, $\mu_k > 0$, be, respectively, the scattering matrix, normalization matrices, and eigenvalues of some (uniquely determined by them) boundary-value problem (A) –(B) with a potential matrix $V(x)$ satisfying condition (2).

Let us note that the indicated conditions are independent. However, if the matrix $S(\lambda)$ has properties I_S, III_S, and, moreover, is continuous at zero, then, denoting by m_- and m_+ the number of linearly independent solutions, respectively, of equations (10) and (12), and by q the rank of the matrix $I - S(0)$, we shall have

$$m_+ - m_- = \frac{\eta(+\infty) - \eta(0)}{2\pi} - \frac{1}{2}q, \tag{15}$$

where $i\eta(\lambda) = \ln(\det S(\lambda))$. Therefore, if $S(\lambda)$ is continuous, has properties I_S, III_S, and the sum of the ranks of the matrices M_k is equal to the right-hand side of formula (15), then properties II_S and V follow from property IV.

5. If a unitary matrix possessing properties I_S and II_S is given, then it is always possible, and in infinitely many ways, to specify numbers $\mu_k > 0$ and Hermitian matrices M_k so that all the conditions I_S, II_S, IV, and V are satisfied. In particular, the following theorem is valid.

Theorem 7. *Properties I_S, II_S, and III_S are necessary and sufficient in order that a unitary matrix $S(\lambda)$ be the scattering matrix of some boundary-value problem (A)–(B) with a potential matrix satisfying condition (2).*

In conclusion, let us note that, proceeding from the results set forth in the present note, one can carry out an analogous investigation also in the case when the elements of the potential matrix contain singularities of order x^{-2} (both at zero and at infinity).

Kharkov State University
named after A. M. Gorky

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Note: Figure translations are in progress. See original paper for figures.

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