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INVESTIGATION OF SUPERSONIC FLOWS BY MEANS OF THE VOLTERRA INTEGRAL

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Abstract

Full Text

HYDROMECHANICS

A. I. UTKIN

**INVESTIGATION OF SUPERSONIC FLOWS
BY MEANS OF THE VOLTERRA INTEGRAL**

(Presented by Academician L. I. Sedov, 16 IV 1957)

The Volterra method ^(1,2) is applied to the investigation of the linear problem of supersonic flow past axisymmetric surfaces. By a series of transformations the Volterra integral is reduced to a linear integral equation of the second order, the solution of which can be obtained by the methods of operational calculus or by series.

As a practical application, for a particular case, the problem proposed by L. I. Sedov concerning the profile of an annular wing of minimum wave drag is considered, and the shape of the generating curve of the bottom region is also determined if the pressure in it is prescribed.

General solution. Consider the flow past an axisymmetric surface S that differs little from a circular cylinder (Fig. 1). If we put

$$\bar{z} = \frac{z}{\sqrt{M_\infty^2 - 1}}, \tag{1}$$

then the equation for the potential of the additional velocities $\Phi(x, y, z)$ will be

$$-\Phi_{\bar{z}\bar{z}} + \Phi_{xx} + \Phi_{yy} = 0.$$

The boundary condition on the surface of the body will be, as usual,

$$\left(\frac{\partial\Phi}{\partial n}\right)_S = V_0\delta,$$

where V_0 is the velocity in the undisturbed flow, δ is the angle between the tangent to the contour and the z -axis. We shall restrict the length of the surface by the condition

$$L_0 < D_0\sqrt{M_\infty^2 - 1},$$

where D_0 is the mean diameter of the surface S .

The Volterra integral ⁽¹⁾ makes it possible to find the value of the potential at any point of the flow $\Phi(x_1, y_1, \bar{z}_1)$, if the values of $\Phi(x, y, \bar{z})$ and $d\Phi/dn$ are known on some surface Σ carrying the initial Cauchy data, n being the normal to the surface ^(1,2).

The Volterra integral has the form

$$\Phi(x_1, y_1, \bar{z}_1) = \frac{1}{2\pi} \frac{\partial}{\partial \bar{z}_1} \iint_{\Sigma} \left(V \frac{d\Phi}{dN} - \Phi \frac{dV}{dN} \right) d\Sigma. \quad (2)$$

The surface $\Sigma = \Sigma_1 + \Sigma_2$ is cut off by the characteristic cone on the initial surface of disturbance Σ_2 and on the body surface Σ_1 (Fig. 1).

Let us set $\Phi = 0$ on the surface Σ_2 ; then $d\Phi/dN$ on Σ_2 also vanishes, since the conormal in this case coincides with the generator of the conical surface Σ_2 , where $\Phi = 0$. Thus, the integration need be carried out only over the surface of the body, i.e. $\Sigma = \Sigma_1$.

In expression (2)

$$V = \ln \frac{(\bar{z}_1 - \bar{z}) - \sqrt{(\bar{z}_1 - \bar{z})^2 - (x_1 - x)^2 - (y_1 - y)^2}}{\sqrt{(x_1 - x)^2 + (y_1 - y)^2}}$$

is Volterra's fundamental function ^(1,2).

Introduce cylindrical coordinates $x = r \cos \varphi$; $y = r \sin \varphi$; $\bar{z} = z$. Differentiating expression (2) with respect to \bar{z}_1 , taking into account the dependence of the limits of integration on \bar{z}_1 , and carrying out a number of transformations, we obtain

$$\Phi(\bar{z}_1, r_1) \left(1 - \frac{1}{2} \sqrt{\frac{2r_1}{D_0}} \right) = -\frac{1}{\pi} \sqrt{\frac{D_0}{2r_1}} \int_0^{z_1 - (r_1 - D_0/2)} \frac{d\Phi}{dn} K(\lambda_1) dz - \frac{1}{\pi} \int_0^{z_1 - (r_0 - D_0/2)} \Phi(\bar{z}_1, D_0) \left[\frac{K(\lambda)}{D_0} + \lambda \frac{\partial K(\lambda)}{\partial z_1} \right] dz \quad (3)$$

where

$$\lambda = \frac{\bar{z}_1 - \bar{z}}{D_0},$$

and $K(\lambda)$ is the complete elliptic integral of the first kind ⁽⁶⁾.

The expression obtained does not make it possible to find the solution directly, since the value of Φ on the surface Σ_1 is unknown. To determine $\Phi(\bar{z}, D_0)$ we use equality (3), putting in it $D_1 \rightarrow D_0$, i.e. we shall consider the point M (Fig. 1) on the surface S ; then equality (3) becomes the integral equation

$$\Phi(\bar{z}_1, D_0) = -\frac{2}{\pi} \int_0^{\bar{z}_1} \frac{d\Phi}{dn} K(\lambda) d\bar{z} - \frac{2}{\pi} \int_0^{\bar{z}_1} \Phi(\bar{z}, D_0) \left[\frac{K(\lambda)}{D_0} + \lambda \frac{\partial K(\lambda)}{\partial z_1} \right] d\bar{z}. \quad (4)$$

The integral equation (4) is a linear integral equation of the second kind—a Volterra equation.

If we put $D_0 \rightarrow \infty$ and pass to Cartesian coordinates, then the last integral in equation (4) vanishes, and the first term, on the basis of expression (2), can be represented in the form of the well-known formula ^(2,3) for a flat wing:

$$\Phi(x, z) = -\frac{1}{\pi} \iint_{\Sigma} \frac{d\Phi}{dn} \frac{d\eta d\xi}{\sqrt{(x-\xi)^2 - (M_\infty^2 - 1)(z-\eta)^2}}.$$

The solution of equation (4) is easily obtained by operational methods. Denote

$$F(p) \doteq \Phi(\bar{z}_1); \quad R(p) \doteq -\frac{2}{\pi} \int_0^{\bar{z}_1} \frac{d\Phi}{dn} [K(\lambda)] d\bar{z};$$

$$Q(p) \doteq \left[\frac{1}{D_0} K(\lambda) \frac{\partial K(\lambda)}{\partial \lambda} \right],$$

where $F(p)$, $R(p)$, and $Q(p)$ are the “images” of the indicated “originals.” Taking into account that the kernel of the integral equation will be symmetric, we obtain ⁽⁴⁾

$$F(p) = \frac{pR(p)}{p - Q(p)};$$

passing again to the “originals,” we find the solution. We note that the solution cannot be written in the form of a finite combination of elementary functions, since the kernel of the integral equation is expressed in terms of elliptic integrals.

The solution of the equations may also be sought in the form of a series

$$\Phi(\bar{z}_1) = \Phi_0(\bar{z}_1) + \eta \Phi_1(z_1) + \eta^2 \Phi_2(z_2) + \dots,$$

where

$$\eta = -2/\pi,$$

$$\Phi_0(\bar{z}_1) = -\frac{2}{\pi} \int_0^{\bar{z}_1} \frac{d\Phi}{dn} K(\lambda) d\bar{z},$$

Fig. 2

Figure 1: Fig. 2

$$\Phi_{n+1}(\bar{z}_1) = \int_0^{\bar{z}_1} \left[\frac{K(\lambda)}{D_0} + \lambda \frac{\partial K(\lambda)}{\partial \bar{z}_1} \right] \Phi_n(\bar{z}) d\bar{z}.$$

The indicated series converges uniformly in z_1 and in η for finite η and for $\bar{z}_1 < D_0$ ⁽⁵⁾. Both methods rapidly lead to the goal in the cases considered by us.

As a particular case of the problem of the profile of an annular wing of minimum wave drag, let us consider an internal triangular profile with a constant value $c/L_0 = \bar{c}$, and let us find the optimal position of its vertex $\bar{a} = a/\sqrt{M_\infty^2 - 1}$ (Fig. 2). On the surface AB the potential, if terms of higher order of smallness are neglected, beginning with $\frac{1}{16} \bar{z}^4/D_0^4$, as follows from the solution of equation (4), will be

$$\Phi = \delta V_0 \bar{z} \sum_{n=0}^3 \frac{1}{2^n} \left(\frac{\bar{z}}{D_0} \right)^n,$$

where $\delta = c/a$.

Fig. 2

The potential on the surface BD , as can be determined using equation (4), in the same approximation has the form

$$\begin{aligned} \Phi_2 = & -\gamma V_0 \bar{\xi} \sum_{n=0}^3 \frac{1}{2^n} \left(\frac{\bar{\xi}}{D_0} \right)^n + \\ & + \delta V_0 \left[Q_0 \left(1 + \frac{\bar{\xi}}{D_0} + \frac{\bar{\xi}^2}{2D_0^2} + \frac{5}{12} \frac{\bar{\xi}^3}{D_0^3} \right) + \frac{1}{12} \frac{\bar{z}^3 - \bar{\xi}^3}{D_0^3} \right], \end{aligned}$$

where

$$\gamma = \frac{c}{L_0 - \bar{a}}, \quad \bar{\xi} = \bar{z} - \bar{a}, \quad Q_0 = \sum_{n=1}^3 \frac{1}{n!} \frac{\bar{a}^n}{D_0^{n-1}}.$$

The polynomials written above are the initial terms of series that converge rapidly absolutely and uniformly (for $\bar{z}/D_0 \leq 1$), since they have been obtained by integrating series expressing elliptic integrals.

Knowing the potential, we find the pressure and then, by integration, the wave-drag coefficient (referred to the area $\pi D_0 L_0$):

Fig. 3

Figure 2: Fig. 3

$$C_x = \frac{8\bar{c}^2}{\sqrt{M_\infty^2 - 1}} \left(1 - \frac{\bar{L}_0 - 2\bar{a}}{4D_0} - \frac{\bar{L}_0^2 - \bar{a}^2}{4D_0^2} + \frac{\bar{a}\bar{L}_0}{8} - \dots \right).$$

Obviously, C_x will be smallest if $\bar{a}/D_0 \rightarrow 0$ and $\bar{L}_0/D_0 \rightarrow 1$, and will be approximately half the wave-drag coefficient of a triangular symmetric wing profile of infinite span.

Fig. 3

From physical considerations it is clear that \bar{a} cannot be very small, but decreasing \bar{a} gives a noticeable gain in drag and is advantageous up to certain limits (possibly up to the formation of a detached shock wave).

Let us consider, as the next application, the shape of the generatrix of the base region $r = f(z)$ (Fig. 3). The dependence $r = f(z)$ can be found by means of equation (4), if the pressure coefficient in the base region $\bar{P} = \text{const}$ is specified. From the condition $\bar{P} = \text{const}$ and equation (4) it follows that along the boundary of the region:

$$\begin{aligned} \frac{\Phi}{V_0} = \bar{P} \frac{\sqrt{M^2 - 1}}{2} \bar{z}_1 = -\frac{2}{\pi} \int_0^{\bar{z}_1} \frac{\partial r}{\partial z} K(\lambda) d\bar{z} - \\ - \frac{2}{\pi} \int_0^{\bar{z}_1} \bar{P} \frac{\sqrt{M^2 - 1}}{2} \left[\frac{K(\lambda)}{D_0} + \lambda \frac{\partial K(\lambda)}{\partial z_1} \right] d\bar{z}, \end{aligned} \quad (5)$$

where, evidently, it is required to find dr/dz . Consequently, the equation written down is a linear integral equation of the first kind. Differentiating (5) with respect to \bar{z}_1 term by term, we pass to a linear integral equation of the second kind; solving it by the methods presented above, we find

$$\frac{dr}{dz} = f(z)$$

and, integrating, finally obtain the equation of the generatrix of the base region (in the linear formulation):

$$r = \frac{D_0}{2} - |\bar{P}| \frac{M^2 - 1}{2} \left(\bar{z} + \frac{\bar{z}^2}{2D_1} - \frac{1}{12} \frac{\bar{z}^3}{D_0^2} + \frac{1}{24} \frac{\bar{z}^4}{D_0^3} - \dots \right).$$

The series obtained converges absolutely and uniformly as the integral of a series expressing the sum of elliptic integrals. By substitution in (5) one verifies the correctness of the solution.

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