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Abstract

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B. M. PERETYAGIN

ON THE NUMBER OF LIMIT CYCLES OF THE EQUATION

$$\frac{dy}{dx} = \frac{cx + dy + P(x,y)}{ax + by + Q(x,y)},$$

where $P(x,y)$ and $Q(x,y)$ are homogeneous polynomials of degree n

(Presented by Academician I. G. Petrovskii, February 14, 1957)

The paper considers the question of the maximal number of limit cycles for the differential equation

$$\frac{dy}{dx} = \frac{cx + dy + P(x,y)}{ax + by + Q(x,y)} \left(\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0 \right), \tag{A}$$

where $P(x,y)$ and $Q(x,y)$ are homogeneous polynomials of degree n, and the origin is a singular point of the second group. Results on this question for equation (A), when $P(x,y)$ and $Q(x,y)$ are polynomials of degree 2, were obtained by N. N. Bautin (^{3}) and by I. G. Petrovskii and E. M. Landis (^{1}). N. F. Otrokov (^{4}) obtained results on the lower bound of the maximal number of limit cycles of equation (A), when $P(x,y)$ and $Q(x,y)$ are polynomials of degree n.

By a linear transformation, equation (A) can always be reduced to the form

$$\frac{dy}{dx} = \frac{-x - a_1 y + \sum\limits_{i+j=n} a_{ij} x^{i y^j}}{y - a_1 x + \sum\limits_{i+j=n} b_{ij} x^{i y^j}}$$

and, passing to polar coordinates, to the form

$$\frac{d\rho}{d\varphi} = \frac{\rho a_1 + \rho^n \alpha_{n-1}}{1 - \rho^{n-1} \beta_{n-1}},$$

where α_{n-1} and β_{n-1} are homogeneous polynomials of degree $n+1$ with respect to $\cos\varphi$ and $\sin\varphi$.

For further computations we pass from trigonometric functions to exponential ones by Euler’ s formulas. Then

$$\alpha_{n-1} = \frac{1}{2^{n+1}} \left[a_{n+1}e^{(n+1)i\varphi} + a_{n-1}e^{(n-1)i\varphi} + \ldots + \bar{a}_{n-1}e^{-(n-1)i\varphi} + \bar{a}_{n+1}e^{-(n+1)i\varphi}\right],$$

$$\beta_{n-1} = \frac{1}{2^{n+1}} \left[b_{n+1}e^{(n+1)i\varphi} + b_{n-1}e^{(n-1)i\varphi} + \ldots + \bar{b}_{n-1}e^{-(n-1)i\varphi} + \bar{b}_{n+1}e^{-(n+1)i\varphi}\right]$$

(a_k and $\bar{a}_k$, respectively b_m and $\bar{b}_m$, are mutually conjugate complex numbers of the form $A'_k \pm iA''_k$ and $B'_m \pm iB''_m$; A'_k, A''_k, B'_m, B''_m are linear homogeneous polynomials in the coefficients of equation (A).

Making the substitution $\rho^{n-1} = \rho_1$ and $(n-1)\varphi = \varphi_1$ and omitting the indices, we finally obtain

$$\frac{d\rho}{d\varphi} = \frac{\rho a_1 + \rho^2 a_{n-1}}{1 - \rho\beta_{n-1}} \tag{1}$$

or

$$\frac{d\rho}{d\varphi} = \rho R_1 + \rho^2 R_2 + \cdots = \sum_{i=1}^{\infty} \rho^i R_i \tag{2}$$

in the region $(0 \le \rho \le \rho_1;\ 0 \le \varphi \le 2\pi)$.

We seek a solution of equation (2), $\rho = \rho(\varphi)$, satisfying the initial condition $\rho_0 = \rho(0)$, in the form of a series in powers of ρ_0:

$$\rho = \rho_0\vartheta_1(\varphi, a_{ij}, b_{ij}) + \rho_0^2\vartheta_2(\varphi, a_{ij}, b_{ij}) + \cdots, \tag{3}$$

converging in the region (G) $(0 \le \varphi \le 2\pi;\ 0 \le \rho_0 \le \rho_2 \le \rho_1)$.

To determine the coefficients $\vartheta_k(\varphi, a_{ij}, b_{ij})$ we have the following recurrence equations

$$\begin{aligned}
&\vartheta'_1 = R_1\vartheta_1,\\
&\vartheta'_2 = R_1\vartheta_2 + R_2\vartheta_1^2,\\
&\quad \ldots\ldots\ldots\ldots\ldots\\
&\vartheta'_k = R_1\vartheta_k + R_2 \sum_{i+j=k} \vartheta_i\vartheta_j + R_3\\
&\qquad \times \sum_{i+j+l=k} \vartheta_i\vartheta_j\vartheta_l + \cdots + R_k\vartheta_1^k.
\end{aligned} \tag{4}$$

Having successively determined the coefficients $\vartheta_k(\varphi, a_{ij}, b_{ij})$ from (4) and substituting $\varphi = 2\pi$ in the solution (3), we obtain the successor function

$$\rho(2\pi) = \rho_0 \vartheta_1(2\pi, a_{ij}, b_{ij}) + \rho_0^2 \vartheta_2(2\pi, a_{ij}, b_{ij}) + \cdots, \tag{5}$$

whose positive roots of the difference with the initial value ρ_0

$$\rho - \rho_0 = \rho_0[\vartheta_1(2\pi, a_{ij}, b_{ij}) - 1] + \rho_0^2 \vartheta_2(2\pi, a_{ij}, b_{ij}) + \cdots, \tag{6}$$

correspond to limit cycles.

The main difficulty of this method consists in establishing the structure of the coefficients of the successor function (5). N. N. Bautin, in order to establish the form of these coefficients for equation (A), when $P(x, y)$ and $Q(x, y)$ are polynomials of degree 2, used the necessary and sufficient conditions for the existence of a center for this equation, the establishment of which is a difficulty of the same, if not greater, order. At the present time the necessary and sufficient conditions are not known for equation (A) even, for example, when $P(x, y)$ and $Q(x, y)$ are polynomials of degree 3.

We shall first determine the form of the coefficients of the successor function for equation (1), in which $a_1 = 0$.

Lemma 1. The coefficients $\vartheta_k(2\pi, a_{ij}, b_{ij})$ are homogeneous polynomials of degree $k-1$ with respect to the coefficients of the functions α_{n-1} and β_{n-1}, and, consequently, of the coefficients of the original differential equation.

Theorem 1. The coefficients $\vartheta_k(2\pi, a_{ij}, b_{ij})$ of the successor function for the differential equation

$$\frac{d\rho}{d\varphi} = \frac{\rho^2 \alpha_{n-1}}{1 - \rho\beta_{n-1}} \tag{7}$$

for even n can be represented in the form:

$$\vartheta_1(2\pi, a_{ij}, b_{ij}) = 1; \qquad \vartheta_2(2\pi, a_{ij}, b_{ij}) = 0; \qquad \vartheta_3(2\pi, a_{ij}, b_{ij}) = \sum \overline{\vartheta}_i^{(3)},$$

$$\vartheta_4(2\pi, a_{ij}, b_{ij}) = \sum \overline{\vartheta}_i^{(3)} \theta_4^{(3)}; \quad \vartheta_5(2\pi, a_{ij}, b_{ij}) = \sum \overline{\vartheta}_i^{(5)} + \sum \overline{\vartheta}_i^{(3)} \theta_{i,5}^{(3)}; \ldots$$

$$\ldots; \quad \vartheta_{2n+1}(2\pi, a_{ij}, b_{ij}) = \sum \overline{\vartheta}_i^{(2n+1)} + \sum \overline{\vartheta}_i^{(2n-1)} \theta_{i,2n+1}^{(2n-1)} + \ldots + \sum \overline{\vartheta}_i^{(3)} \theta_{i,2n+1}^{(3)};$$

$$\vartheta_m(2\pi, a_{ij}, b_{ij}) = \sum \overline{\vartheta}_i^{(2n+1)} \theta_{i,m}^{(2n+1)} + \sum \overline{\vartheta}_i^{(2n-1)} \theta_{i,m}^{(2n-1)} + \ldots + \sum \overline{\vartheta}_i^{(3)} \theta_{i,m}^{(3)}$$

for all $m > 2n+1$, where $\overline{\vartheta}_i^{(k)}$ are homogeneous polynomials in the coefficients of the equation of degree $k-1$; $\theta_{i,l}^{(k)}$ are homogeneous polynomials in the same coefficients of degree $l-k$, which we shall call supplements.

Proof. $\vartheta_1(2\pi, a_{ij}, b_{ij}) = 1$ follows from the first equation of system (4). $\vartheta_2(2\pi, a_{ij}, b_{ij}) = 0$, since $\vartheta_2(\varphi, a_{ij}, b_{ij})$ contains only periodic terms. $\vartheta_3(2\pi, a_{ij}, b_{ij})$ will contain all possible products of zero weight* of the 2nd order of the form $a_k \overline{b}_k$ and their conjugates:

$$\vartheta_3(2\pi, a_{ij}, b_{ij}) = c_1(a_{n-1}\overline{b}_{n-1} + \overline{a}_{n-1}b_{n-1}) + c_2(a_{n-3}\overline{b}_{n-3} - \overline{a}_{n-3}b_{n-3}) + \ldots$$

$$\ldots + c_{n/2}(a_1\overline{b}_1 + \overline{a}_1 b_1) = \sum \overline{\vartheta}_i^{(3)}.$$

$$\vartheta_4(2\pi, a_{ij}, b_{ij}) = \sum \overline{\vartheta}_i^{(3)} \theta_4^{(3)}.$$

And in general, it is not difficult to observe that all even coefficients contain no new products of zero weight; therefore they can be expressed through the preceding odd ones, and in what follows we shall not compute them. $\vartheta_5(2\pi, a_{ij}, b_{ij})$ consists of the sum of all possible simple and composite products of zero weight of the 4th order. Composite products of the 4th order can be written in the form $\sum \overline{\vartheta}_i^{(3)} \theta_{i,5}^{(3)}$, while simple products we denote by $\sum \overline{\vartheta}_i^{(5)}$. Then

$$\vartheta_5(2\pi, a_{ij}, b_{ij}) = \sum \overline{\vartheta}_i^{(5)} + \sum \overline{\vartheta}_i^{(3)} \theta_{i,5}^{(3)}.$$

Continuing this process of computation, we arrive at the highest simple products of zero weight, which will be products of the form

$$a_{n+1}^{n-1} \sum_{i+j=n+1} \overline{a}_{n-1}^{\,i} \overline{b}_{n-1}^{\,j}$$

and their conjugates, of order $(n-1)+(n+1) = 2n$. On the basis of Lemma 2 this order corresponds to the coefficient $\vartheta_{2n+1}(2\pi, a_{ij}, b_{ij})$. Thus,

$$\vartheta_{2n+1}(2\pi, a_{ij}, b_{ij}) = \sum \overline{\vartheta}_i^{(2n+1)} + \sum \overline{\vartheta}_i^{(2n-1)} \theta_{i,2n+1}^{(2n-1)} + \ldots + \sum \overline{\vartheta}_i^{(3)} \theta_{i,2n+1}^{(3)}.$$

All remaining products of zero weight of order $m > 2n$ will be composite, and they can be expressed through the existing simple products,

$$\vartheta_m(2\pi, a_{ij}, b_{ij}) = \sum \overline{\vartheta}_i^{(2n+1)} \theta_{i,m}^{(2n+1)} + \sum \overline{\vartheta}_i^{(2n-1)} \theta_{i,m}^{(2n-1)} + \ldots + \sum \overline{\vartheta}_i^{(3)} \theta_{i,m}^{(3)}.$$

For $n = 2k+1$ odd, the form of the coefficients of the successor function is established analogously.

Theorem 2. The coefficients of the successor function for $a_1 \neq 0$ have the form

$$\vartheta_k(2\pi, a_{ij}, b_{ij}) = \vartheta_k^*(2\pi, a_{ij}, b_{ij}) + a_1\theta_k^{(1)},$$

where $\vartheta_k^*(2\pi, a_{ij}, b_{ij})$ are the coefficients of the successor function of equation (1), in which $a_1 = 0$.

The assertion follows from the form of system (4) and the form of R_k.

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* Products of coefficients $(a_k \dots b_l \dots \bar{a}_m \dots \bar{b}_p)$, the sum of whose indices $[k + \dots + l + \dots + (-m) + \dots + (-p)]$, giving the coefficient of $i\varphi$ in the exponent of e, is equal to zero, will be called products of zero weight. If this product cannot be represented as a product of two products of zero weight, we shall call it simple; otherwise, composite. These concepts were introduced in (2).

Theorem 3. *The number of limit cycles that appear when the coefficients of equation (1) are varied in the domain* (G) *does not exceed* $n+1$*. One can indicate a point of the coefficient space of the equation such that in an* ε*-neighborhood of it and in a* δ*-neighborhood of the origin, equation (1) has* $(n+2)/2$ *limit cycles for even* n *and* $(n+3)/2$ *limit cycles for odd* n*.*

For the proof, taking into account the form of the coefficients, for example for even n, the difference (6) can be written in the form

$$\rho(2\pi) - \rho_0 = \rho_0 \left[a_1 z^{(1)} + \rho_0^2 \sum \bar{\vartheta}_i^{(3)} z_i^{(3)} + \cdots + \rho_0^{2n} \sum \bar{\vartheta}_i^{(2n+1)} z_i^{(2n+1)} \right]. \qquad (8)$$

Here

$$z_i^{(k)} = 1 + \rho_0 A_{i,k+1}^{(k)} + \rho_0^2 A_{i,k+2}^{(k)} + \cdots.$$

Finally, taking into account the form of the complements $\vartheta_{i,m}^{(k)}$, the difference (8) can be written in the form

$$\begin{aligned} \rho - \rho_0 = \rho_0 [& a_1 \psi_1 + \rho_0^2 \bar{\vartheta}_3 \psi_3 + \rho_0^4 (\bar{\vartheta}_5 \psi_5 + \bar{\vartheta}_5^* \psi_5^*) \\ & + \rho_0^6 (\bar{\vartheta}_7 \psi_7 + \bar{\vartheta}_7^* \psi_7^*) + \cdots \\ & + \rho_0^{2n} (\bar{\vartheta}_{2n+1} \psi_{2n+1} + \bar{\vartheta}_{2n+1}^* \psi_{2n+1}^*) + \rho_0^{2n+2} \bar{\vartheta}_{2n+3}^* \psi_{2n+3}^*], \end{aligned}$$

where

$$\psi_k = 1 + \rho_0 D_{k+1}^{(k)} + \rho_0^2 D_{k+2}^{k} + \cdots; \qquad \psi_k^* = 1 + \rho_0 D_{k+1}^{*(k)} + \rho_0^2 D_{k+2}^{*(k)} + \cdots.$$

It is not difficult to see that the maximum number of positive roots for the function (8), and consequently also of limit cycles for equation (1), does not exceed $n+1$.

To prove the second part of the theorem, consider the point A_0 of the coefficient space E_m $(m = (n-1)(n+4)+1)$, at which

$$a_1 = 0; \quad a_{n+1} = a_{n+1}^*; \quad b_1 = b_1^*; \quad b_k = 0 \quad (k = 3, 5, \dots, n-1); \quad a_m = 0 \quad (m = 1, 3, 5, \dots, n-1),$$

and consequently all $\bar{\vartheta}_i^{(k)}$, except

$$\bar{\vartheta}^{(n+3)} = c_{n+3}(a_{n+1}\bar{b}_1^{n+1} + \bar{a}_{n+1}b_1^{n+1}),$$

vanish. Varying in an ε-neighborhood of the point A_0 the coefficients $a_{n-1}, a_{n-3}, \dots, a_1$, it is not difficult to show that in this case the difference (8)

$$\rho(2\pi) - \rho_0 = \rho_0 \left[a_1\psi_1 + \rho_0^2\bar{\vartheta}^{(3)}\psi_3 + \rho_0^4\bar{\vartheta}^{(5)}\psi_5 + \dots + \rho_0^{n+2}\bar{\vartheta}^{(n+3)}\varphi_{n+3}\right]$$

changes sign $(n+2)/2$ times on the interval $(0, \delta)$, whence follows the existence, for equation (1), of at least $(n+2)/2$ limit cycles.

Smolensk State Pedagogical Institute
named after Karl Marx

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Note: Figure translations are in progress. See original paper for figures.