MATHEMATICS

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1957

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Abstract

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ON ONE CONDITION FOR THE UNRESTRICTED APPLICABILITY OF S. A. CHAPLYGIN' S THEOREM ON INEQUALITIES TO SYSTEMS OF FIRST-ORDER DIFFERENTIAL EQUATIONS

(Presented by Academician S. L. Sobolev on 31 V 1957)

For a single first-order differential equation there is Chaplygin' s theorem on inequalities:

If a function $u = u(x)$ is such that $u' - f(x, u) > 0$ (< 0) on the segment $[x_0, x_1]$ and $u(x_0) = y_0$, then for the solution $y = y(x)$ of the equation $y' = f(x, y)$, $y(x_0) = y_0$, the inequality $y(x) < u(x)$ $(y(x) > u(x))$ holds on $(x_0, x_1]$. (1).

This theorem does not extend directly to systems of differential equations. This may be seen from the following example: $y_1' - y_1 + y_2 - 2x - 1 = 0$, $y_2' - y_2 + y_1 - 2x + 1 = 0$, $y_1(0) = 0$, $y_2(0) = 1$. The solution of the system is $y_1 = x^2$, $y_2 = 1 + x^2$. As comparison functions let us take $u_1 = \frac{x}{5}$ and $u_2 = 1 + 3x$. We have $u_1' - u_1 + u_2 - 2x - 1 = \frac{1 + 4x}{5} > 0$, $u_2' - u_2 + u_1 - 2x + 1 = 3 - \frac{24}{5}x > 0$ for $x < {}^5/_8$. But to the right of the point $x = {}^1/_5$, $u_1 - y_1 = ({}^1/_5 - x)x < 0$, despite the fulfillment of the differential inequalities.

Thus, if we have a system of equations

$$y_i' = f_i(x, y_1, y_2, \dots, y_n) \tag{1}$$

with initial conditions

$$y_i(x_0) = y_{i0}, \tag{2}$$

then, in order that from the relations $u_i' - f_i(x, u_1, u_2, \dots, u_n) > 0$ (< 0), $u_i(x_0) = y_{i0}$, it should follow that $u_i(x) > y_i(x)$ $(u_i(x) < y_i(x))$, it is necessary to impose additional conditions either on the right-hand sides of the equations of the system (3), or on the comparison functions $u_1, u_2, \dots, u_n$.

In the present paper one of the possible conditions imposed on the comparison functions is indicated.

First we record several auxiliary propositions. In doing so we shall suppose that the functions $f_i(x, y_1, y_2, \dots, y_n)$ are continuous in some domain D of variation of their arguments and satisfy in this domain the Lipschitz condition with constant K with respect to the arguments $y_1, y_2, \dots, y_n$. In addition, we assume that for $x \in [x_0, x_1]$ the values of all introduced comparison functions do not leave the domain D.

Lemma 1. *If for the functions $u_i = u_i(x)$, $\vartheta_i = \vartheta_i(x)$ $(i = 1, 2, \dots, n)$ the conditions are satisfied:*

1) $u_i(x_0) = \vartheta_i(x_0) = y_{i0}$;
2) $u_i' - f_i(x, s_1, s_2, \dots, s_n) \le 0$, $\vartheta_i' - f(x, s_1, s_2, \dots, s_n) \ge 0$ for $x \in [x_0, x_1]$ and any s_i lying between $u_i(x)$ and $\vartheta_i(x)$,

then on the entire segment $[x_0, x_1]$ $u_i \le y_i \le \vartheta_i$, where $y_1, y_2, \dots, y_n$ is the solution of system (1) satisfying the initial conditions (2).

Lemma 2. Let, for the functions $u_i = u_i(x)$, $i = 1, 2, \dots, n$, the initial conditions (2) be satisfied; let $\eta_1, \eta_2, \dots, \eta_n$ be a solution of the system

$$\eta_i' - K \sum_{r=1}^{n} \eta_r = \left| u_i' - f_i(x, u_1, \dots, u_n) \right|, \tag{3}$$

satisfying zero initial conditions. Then on $[x_0, x_1]$ the values of the functions $u_i + \eta_i$ $(u_i - \eta_i)$ are not less than (not greater than) the corresponding values of the functions $y_1, y_2, \dots, y_n$ giving a solution of system (1) for the same initial conditions, i.e. on $[x_0, x_1]$

$$u_i - \eta_i \le y_i \le u_i + \eta_i.$$

For the proof, let us note that $\eta_i(x) \ge 0$ on $[x_0, x_1]$ (this can be verified by using the results of the work ([3])). Consequently,

$$u_i - \eta_i \le u_i \le u_i + \eta_i.$$

Consider the difference $(u_i + \eta_i)' - f_i(x, s_1, s_2, \dots, s_n)$, where, for the given x, s_r assumes arbitrary values lying in the segment $[u_r(x) - \eta_r(x),\ u_r(x) + \eta_r(x)]$.

We have:

$$(u_i+\eta_i)'-f_i(x,s_1,s_2,\dots,s_n)=$$
$$=u_i'-f_i(x,u_1,u_2,\dots,u_n)+|u_i'-f_i(x,u_1,u_2,\dots,u_n)|+\eta_i'-K\sum_{r=1}^{n}\eta_r-$$
$$-|u_i'-f_i(x,u_1,u_2,\dots,u_n)|+f_i(x,u_1,u_2,\dots,u_n)-f_i(x,s_1,s_2,\dots,s_n)+$$
$$+K\sum_{r=1}^{n}\eta_r\geq K\sum_{r=1}^{n}\{\eta_r-|u_r-s_r|\}\geq 0,$$

since $|u_r-s_r|\leq\eta_r$. Similarly we prove that

$$(u_i-\eta_i)'-f_i(x,s_1,s_2,\dots,s_n)\leq 0$$

for the same $s_1,s_2,\dots,s_n$. It remains to apply Lemma 1.

Lemma 3. If

$$\eta_i'-K\sum_{r=1}^{n}\eta_r=\varepsilon_1(x),\qquad \sigma_i'-K\sum_{r=1}^{n}\sigma_r=\varepsilon_2(x),$$

$\sigma_i(x_0)=\eta_i(x_0)$, $i=1,2,\dots,n$, and $\varepsilon_1(x)\leq\varepsilon_2(x)$ on $[x_0,x_1]$, then on the same interval $\eta_i(x)\leq\sigma_i(x)$.

The required conclusion is obtained by applying the result of the work ([3]).

Theorem. Let the functions $u_1,u_2,\dots,u_n$ satisfy the initial conditions (2) and the inequalities

$$u_i'-f_i(x,u_1,u_2,\dots,u_n)\geq 0\ (\leq 0)$$

for $x\in[x_0,x_1]$. Let $\vartheta_i=u_i+\eta_i$ ($\vartheta_i=u_i-\eta_i$), where $\eta_1,\eta_2,\dots,\eta_n$ is a solution of the system

$$\eta_i'-K\sum_{r=1}^{n}\eta_r=|u_i'-f_i(x,u_1,\dots,u_n)|\,,$$

vanishing for $x=x_0$, and let $y_1,y_2,\dots,y_n$ be a solution of system (1) satisfying conditions (2). Then, if

$$|\vartheta_i'-f_i(x,\vartheta_1,\vartheta_2,\dots,\vartheta_n)|\leq|u_i'-f_i(x,u_1,u_2,\dots,u_n)|\,,$$

then

$$u_i \geq y_i \quad (u_i \leq y_i)$$

on the entire segment $[x_0, x_1]$.

By virtue of Lemma 1, the values of the functions $\vartheta_i = u_i + \eta_i$ ($\vartheta_i = u_i - \eta_i$) are not less than (not greater than) the values of the functions y_i.

Further, if

$$\sigma_i' - K \sum_{r=1}^{n} \sigma_r = \left|\vartheta_i' - f_i(x, \vartheta_1, \vartheta_2, \dots, \vartheta_n)\right|, \qquad \sigma_i(x_0) = 0,$$

then the functions $\vartheta_i - \sigma_i$ ($\vartheta_i + \sigma_i$) are not greater than (not less than) the functions y_i, i.e.

$$\vartheta_i - \sigma_i \leq y_i \quad (\vartheta_i + \sigma_i \geq y_i).$$

By Lemma 3 and the hypotheses of the theorem, $\sigma_i \leq \eta_i$; therefore

$$u_i \leq u_i + \eta_i - \sigma_i = \vartheta_i - \sigma_i \leq y_i \quad \left(u_i \geq u_i - \eta_i + \sigma_i = \vartheta_i + \sigma_i \geq y_i\right).$$

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Received
25 X 1957

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Note: Figure translations are in progress. See original paper for figures.