



Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

1957

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Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1957. Volume 114, No. 1

MATHEMATICS

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STABILITY OF SOLUTIONS OF PERIODIC SYSTEMS OF EQUATIONS IN TOTAL DIFFERENTIALS

(Presented by Academician I. G. Petrovsky, 28 XI 1956)

Let there be given a system of equations in total differentials of the form

$$dx = p(u)x du + q(v)x dv, \quad (1)$$

where x is a column matrix; $p(u)$ and $q(v)$ are square matrices, continuous, bounded, and satisfying the integrability condition

$$p(u)q(v) = q(v)p(u) \quad (2)$$

for all $u > 0$, $v \geq 0$ (u, v are parameters).

It is known ⁽¹⁾ that in this case system (1) admits a unique solution corresponding to prescribed initial conditions.

By an **integral matrix** of system (1) we shall mean a square matrix

$$X(u, v) = \|x_{ik}(u, v)\|_1^n,$$

whose columns are n linearly independent solutions of this system. Clearly, the matrix $X(u, v)$ satisfies the equation

$$dX = p(u)X du + q(v)X dv. \quad (3)$$

It is easy to show that if \bar{X} is a nonsingular ($|\bar{X}| \neq 0$) particular solution of equation (3), then the general solution of this equation will be

$$X(u, v) = \bar{X}(u, v)C, \quad (4)$$

where C is an arbitrary constant matrix.

If $X(u, v)$ is the normalized integral matrix of system (1), and $\bar{X}(u)$ and $\bar{X}(v)$ are the normalized integral matrices of the systems

$$dx = p(u)x du, \quad dx = q(v)x dv,$$

then

$$X(u, v) = \bar{X}(u)\bar{X}(v) = \bar{X}(v)\bar{X}(u). \quad (5)$$

Indeed,

$$d(\bar{X}\bar{X}) = d(\bar{X})\bar{X} + \bar{X}d(\bar{X}) = p(u)\bar{X}\bar{X} du + \bar{X}q(v)\bar{X} dv.$$

But since (2)

$$\bar{X}(u) = E + \int_{u_0}^u p(u) du + \int_{u_0}^u p(u) du \int_{u_0}^u p(u) du + \dots,$$

then, taking (2) into account, we conclude that the matrix $\bar{X}\bar{X}$ (as well as the matrix $\bar{X}\bar{X}$) satisfies equation (3). And since

$$X(u_0, v_0) = \bar{X}(u_0)\bar{X}(v_0) = \bar{X}(v_0)\bar{X}(u_0) = E,$$

relation (5) is proved.

It follows from this that every integral matrix of system (1) is nonsingular. Indeed, on the basis of (4) and (5),

$$|X(u, v)| = |\bar{X}| |\bar{X}| |C| = \exp \left\{ \int_{u_0}^u \text{Sp}(p) du + \int_{v_0}^v \text{Sp}(q) dv \right\} |C|. \quad (6)$$

But the determinant $|X(u, v)|$ cannot be identically equal to zero; hence $|C| \neq 0$, and the assertion is proved. Thus, all integral matrices of system (1) are obtained by formula (4) for $|C| \neq 0$.

We shall call system (1) **reducible** if, by means of the linear transformation

$$x = L(u, v)y \quad (7)$$

it is reduced to the form

$$dy = Ay du + By dv, \quad (8)$$

where A and B are constant permutable matrices. With respect to the transformation matrix $L(u, v) = \|l_{ik}(u, v)\|_1^n$ we shall assume that in the domain $u \geq 0, v \geq 0$: 1) $L(u, v)$ has continuous derivatives $\partial L/\partial u, \partial L/\partial v, \partial^2 L/\partial u \partial v$, of which the first, as well as the matrix $L(u, v)$ itself, are bounded; 2) $\text{mod } |L(u, v)| > m > 0$.

The matrix $L(u, v)$ will be called a **Lyapunov matrix**.

It is not difficult to establish that if the zero solution of the system

$$dx = P(u, x) du + Q(v, x) dv, \quad (9)$$

where the matrices P, Q satisfy the known conditions (3), is stable or asymptotically stable, then the zero solution of the system

$$dy = \bar{P}(u, v, y) du + \bar{Q}(u, v, y) dv, \quad (10)$$

into which system (9) passes after the transformation (7), will possess the same property.

Theorem 1. *If $p(u)$ and $q(v)$ are periodic matrices, then system (1) is reducible.*

In proving this theorem we use a lemma analogous to the lemma of N. P. Erugin⁽⁵⁾.

Lemma. *In order that system (1) be reducible to the form (8), it is necessary and sufficient that it have a solution of the form*

$$X(u, v) = L(u, v)e^{Au+Bv}. \quad (11)$$

Corollary 1. In order that the "union"

$$dx = \left(\varphi(t)p \left(\int_0^t \varphi dt \right) + \psi(t)q \left(\int_0^t \psi dt \right) \right) x dt$$

of the periodic systems $dx = p(u)x du, dx = q(v)x dv$ be a reducible system, it is sufficient that the matrices $p(u)$ and $q(v)$ be permutable, and that the functions $\varphi(t), \psi(t)$ have the form $\varphi(t) = \alpha + \varphi_1(t), \psi(t) = \beta + \psi_1(t)$, where α and β are constants, and $\varphi_1(t)$ and $\psi_1(t)$ are continuous functions for which the integrals $\int_0^t \varphi_1(t) dt, \int_0^t \psi_1(t) dt$ are bounded.

Corollary 2. In order that the zero solution of the system

$$dx = \left(\varphi(t)p \left(\int_0^t \varphi dt \right) + \psi(t)q \left(\int_0^t \psi dt \right) \right) x dt +$$

$$+ \left(\varphi(t)P_1 \left(\int_0^t \varphi dt, x \right) + \psi(t)Q_1 \left(\int_0^t \psi dt, x \right) \right) dt,$$

which is the “union” of the systems

$$dx = p(u)x du + P_1(u, x) du, \quad dx = q(v)x dv + Q_1(v, x) dv$$

($p(u)$ and $q(v)$ are periodic matrices), stable in the first approximation, be stable in the first approximation, it is sufficient that the matrices $p(u)$ and $q(v)$ be permutable, and that the constant \mathcal{L} in the inequality

$$\text{mod}(P_1, Q_1) < \mathcal{L} \sum_{s=1}^n |x_s| \quad (12)$$

be sufficiently small ((3), Theorem 7).

Corollary 2 remains valid when the periodic matrices $p(u)$ and $q(v)$ are permutable and the positive numbers α and β satisfy the inequality

$$\max \text{Re}(\alpha\lambda^{(a)}) + \max \text{Re}(\beta\lambda^{(b)}) < 0,$$

where $\lambda^{(a)}$ and $\lambda^{(b)}$ are the eigenvalues of the matrices A and B ((3), Theorem 5).

Let us consider system (9). Suppose it has the form

$$dx = p(u)x du + q(v)x dv + P_1(u, x) du + Q_1(v, x) dv, \quad (13)$$

where $p(u)$ and $q(v)$ are periodic matrices, and the matrices $P_1(u, x)$ and $Q_1(v, x)$ satisfy condition (12).

Theorem 2. *The zero solution of system (13) is asymptotically stable if the characteristic numbers of the systems*

$$dx = p(u)x du, \quad dx = q(v)x dv$$

are positive, and the constant \mathcal{L} in inequality (12) is sufficiently small. In particular, if $p(u)$, $q(v)$ are permutable matrices, then the zero solution of system (1) is also asymptotically stable ((3), Theorem 3).

Corollary. The zero solution of the system

$$dx = \left(\varphi(t)p \left(\int_0^t \varphi dt \right) + \psi(t)q \left(\int_0^t \psi dt \right) \right) x dt +$$

$$+ \left(\varphi(t)P_1 \left(\int_0^t \varphi dt, x \right) + \psi(t)Q_1 \left(\int_0^t \psi dt, x \right) \right) dt$$

is asymptotically stable if $\varphi(t)$ and $\psi(t)$ are continuous, bounded, positive functions for which at least one of the integrals

$$\int_0^\infty \varphi(t) dt, \quad \int_0^\infty \psi(t) dt$$

diverges.

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Received
31 V 1955

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Note: Figure translations are in progress. See original paper for figures.

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