MATHEMATICS

T. A. GERMOGENOVA

1957

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Abstract

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MATHEMATICS

T. A. GERMOGENOVA

ON BOUNDED SOLUTIONS, ON A SEMI-INFINITE INTERVAL, OF AN INHOMOGENEOUS INTEGRAL EQUATION WITH A KERNEL DEPENDING ON THE DIFFERENCE OF THE ARGUMENTS

(Presented by Academician M. V. Keldysh, January 7, 1957)

The solution of the equation

$$f(x) = \int_0^\infty k(x-y) f(y)\, dy + g(x) \tag{1}$$

in the case when $g(x)$ and $k(x)e^{\lambda|x|}$ are absolutely integrable for all $\lambda < 1$ and have bounded variation was investigated by V. A. Fock ([1]). He considered only solutions tending to zero at infinity. In the presence of zero or purely imaginary roots of the characteristic equation of the problem

$$1 - \varkappa(u) = 0, \tag{2}$$

where

$$\varkappa(u) = \int_{-\infty}^{\infty} k(x) e^{-ux}\, dx,$$

for the existence of such solutions it proves necessary that the orthogonality conditions

$$\int_0^\infty g(x) f_r(x)\, dx = 0; \tag{3}$$

be satisfied. Here $f_r(x)$ are solutions of the homogeneous equation that, as $x \to \infty$, grow more slowly than exponentially. For certain equations whose free terms do not satisfy relations (3), but whose kernels $k(x)$ obey special conditions, Hopf ([2]) found solutions that are bounded, rather than tending to zero, as $x \to \infty$.

In the present note the question of the existence of a solution is considered in the general case, when the functions $g(x)$ and $k(x)e^{\lambda|x|}$ satisfy only the condition of square integrability on the infinite interval (the kernel $k(x)$ is assumed symmetric). Such an extension of the class of equations under investigation makes it possible, in constructing the solution by the Wiener-Hopf method ([3]) (or by Fock' s equivalent method), to use more convenient estimates and to simplify the reasoning. The expression obtained for the solution in the form of a complex integral makes it possible to investigate the behavior of the solution at infinity.

Theorem*. *A solution of equation (1) bounded at infinity exists and can be constructed by the Wiener-Hopf method if the kernel $k(x)$ and the free term $g(x)$ of the equation satisfy the following conditions:*

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* The existence of solutions satisfying the condition $f(x)e^{-yx} \in L_2(0,\infty)$ $(y > \alpha > 0)$ was shown by I. M. Rapoport ([4]), who formulated and investigated the corresponding Riemann boundary-value problem. We give a proof of the theorem by the Wiener-Hopf method, since the expression for the solution obtained in the proof is used below to investigate the asymptotic properties of the solution.

1) $g(x)e^{-\varepsilon x}$ and $k(x)e^{\lambda|x|}$, for all $\lambda < 1$ and for at least one $\varepsilon < 0$, are square-integrable on the interval $(0,\infty)$; 2) $k(x) = k(-x)$; 3) the multiplicity of the roots of the characteristic equation (2) lying on the imaginary axis does not exceed two.

In the proof it is more convenient first to construct the solution, assuming the existence of the solution $f(x)$ itself and of its Laplace transform

$$\Phi(u) = \int_0^\infty f(x)e^{-ux}\,dx,$$

and then to show that the constructed solution satisfies equation (1).

Putting $f(x) = 0$ and

$$g(x) = -\int_0^\infty k(x-y)f(y)\,dy$$

for $x < 0$, and introducing the functions

$$\gamma_1(u) = \int_0^\infty g(x)e^{-ux}\,dx, \qquad \gamma_2(u) = \int_{-\infty}^0 g(x)e^{-ux}\,dx,$$

we obtain for $\Phi(u)$ the equation

$$\Phi(u)[1-\chi(u)] = \gamma_1(u) + \gamma_2(u). \tag{4}$$

The function $1-\chi(u)$ is regular in the strip $|\operatorname{Re} u| < 1$; in every inner strip $|\operatorname{Re} u| \leq \beta < 1$ it has a finite number $2n$ of zeros u_ν, and can be represented by the expression

$$1-\chi(u) = \frac{\sigma_+(u)}{\sigma_-(u)} \prod_{\nu=1}^{n} (u^2 - u_\nu^2),$$

where

$$\sigma_+(u) = \tau_+(u)(u+1)^{-n}, \qquad \sigma_-(u) = \tau_-(u)(u-1)^n,$$

$$\ln \tau_-(u) = -\frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \frac{\ln \tau(v)}{v-u}\, dv, \qquad \ln \tau_+(u) = -\frac{1}{2\pi i} \int_{-\beta-i\infty}^{-\beta+i\infty} \frac{\ln \tau(v)}{v-u}\, dv,$$

$$\tau(u) = [1-\chi(u)] \frac{(u^2-1)^n}{\prod_1^n (u^2-u_\nu^2)} = \frac{\tau_+(u)}{\tau_-(u)}.$$

The function $\sigma_+(u)$ is regular and has no zeros in the half-plane $\operatorname{Re} u \geq -\beta$, and $\sigma_-(u)$ in the half-plane $\operatorname{Re} u \leq \beta$; the quantities $|\sigma_+(u) u^n|$ and $|\sigma_-(u) u^{-n}|$ are bounded for large $|u|$ in the corresponding half-planes (3). The function $\gamma_1(u)$ is regular in the half-plane $\operatorname{Re} u > \varepsilon$, and $\gamma_2(u)$, as is easy to show, in the half-plane $\operatorname{Re} u < 1$.

As functions of t, $t = \operatorname{Im} u$, $\gamma_1(u)$ and $\gamma_2(u)$ are square-integrable on the infinite interval in the corresponding domains (Plancherel' s theorem). Therefore, using Cauchy' s integral formula, the quantity $\gamma_1(u)\tau_-(u)$ can be represented as the sum $G_1(u) + G_2(u)$, where the functions

$$G_1(u) = -\frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \gamma_1(v)\tau_-(v) \frac{dv}{v-u}, \qquad G_2(u) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \gamma_1(v)\tau_-(v) \frac{dv}{v-u}$$

are, respectively, regular in the half-planes $\operatorname{Re} u > \varepsilon$ and $\operatorname{Re} u < \beta$ and square-integrable in t $(1 > \beta > \varepsilon > 0)$.

The function $\Phi(u)$ in the half-plane $\operatorname{Re} u > 0$ must be regular and square-integrable in t. If (4) is rewritten in the form

$$\sigma_+(u)\Phi(u) \prod_1^n (u^2 - u_\nu^2) - G_1(u)(u-1)^n =$$

$$= G_2(u)(u-1)^n + \gamma_2(u)\sigma_-(u), \tag{5}$$

it is easy to see that the left- and right-hand sides are regular, respectively, in the half-planes $\operatorname{Re} u > 0$ and $\operatorname{Re} u < \beta$, which have the common strip $\varepsilon < \operatorname{Re} u < \beta$. As $|u| \to \infty$ in the corresponding half-planes, both sides grow no faster than $|u|^n$, and, consequently, relation (5) determines a certain polynomial of degree $n-1$. The assumption of a higher degree n contradicts the square integrability with respect to t of the functions $G_1(u)$ and $\Phi(u)$.

For $\Phi(u)$ from (5) we obtain the expression

$$\Phi(u) = \frac{P_{n-1}(u) + G_1(u)(u-1)^n}{\prod_1^n (u^2 - u_\nu^2)\sigma_+(u)}, \tag{6}$$

and the solution is found by the formula

$$f(x) = \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \Phi(u) e^{ux}\, du. \tag{7}$$

The coefficients of the polynomial $P_{n-1}(u)$ are determined from the condition that the function $\Phi(u)$ have no poles of any order in the half-plane $\operatorname{Re} u > 0$, and no poles of order higher than the first on the imaginary axis. The integral on the right-hand side of (7) exists, since the function $G_1(u)$, as a function of t, is square integrable in the strip $0 < \operatorname{Re} u < \beta < 1$.

Just as in the investigation of the homogeneous equation ([3]), it can be shown that the integral (7) indeed satisfies equation (1).

The requirement that the solution be bounded leads to n conditions: the function $\Phi(u)$ must have no poles in the right half-plane and must not have poles of second order on the imaginary axis. The polynomial $P_{n-1}(u)$, and hence also the solution, are uniquely determined by these conditions.

In the general case, the presence of an m-fold root on the imaginary axis is connected with the existence of m linearly independent solutions which, as $x \to \infty$, grow no faster than x^{m-1}. A solution tending to zero as $x \to \infty$ exists when m additional orthogonality conditions are satisfied, as was shown by Fok ([1]).

Equation (1) has no other solutions besides those obtained in this way. This can be proved by slightly modifying Fok' s proof given for solutions tending to zero at infinity.

Asymptotic behavior of the solution of the nonhomogeneous equation. Suppose that the function $g(x)$ behaves as $e^{-u^* x}$ as $x \to \infty$, and $u^* > \max \operatorname{Re} u_\nu$. Then, in the integral (7), in order to extract the principal part of the solution, the contour of integration may be shifted to the left to $\beta + \sigma$

$$f(x) = \sum_{\nu=1}^{n} Q_\nu(x) e^{-u_\nu x} + \frac{1}{2\pi i} \int_{-\sigma - i\infty}^{-\sigma + i\infty} \Phi(u) e^{ux}\, du, \qquad \max \operatorname{Re} u_\nu < \sigma < u^*, \tag{8}$$

where $Q_\nu(x)$ is a polynomial in x whose degree is lower than the multiplicity of the root u_ν, while the integral on the right-hand side is a quantity of order $e^{-\sigma x}$.

In those cases when the behavior of $g(x)$ as $x \to \infty$ turns out to be more complicated, it is necessary to investigate the nature of the singularities of the function $\gamma_1(u)$ in the strip $-\sigma < \operatorname{Re} u < \beta$.

Hopf' s case. If $k(x)$ is represented by the integral $\int_0^\infty e^{-sx}\,d\rho(s)$, where $\rho(s)$ is an increasing function of s, then the characteristic equation (2) has only two roots ([2]). Let us refine formula (8) in this case. Suppose (equation 2) has two real roots u_1 and $u_{-1} = -u_1$. Then

$P_{n-1}(u)$ in formula (6) is a constant equal to $-G_1(u_1)(u_1-1)$. Introducing the new variable of integration $u=-v$ in the integral for $G_1(u_1)$, we write the expression $-G_1(u_1)(u_1-1) - G_1(-u_1)(u_1+1)$ in the form

$$\frac{1}{2\pi i}\int_{\beta^*-i\infty}^{\beta^*+i\infty} \gamma_1(-u)\tau_-(-u)\frac{1+u}{u^2-u_1^2}\,2u_1\,du \qquad (0<\beta^*<u^*). \tag{9}$$

As shown in (2), the Laplace transform of the solution of the homogeneous equation is $\Phi_0(u) = \dfrac{c_0}{u^2-u_1^2}\dfrac{1+u}{\tau_+(u)}$, and since $\tau_-(-u) = \dfrac{1}{\tau_+(u)}$, the integral (9) is equal to

$$\frac{u_1}{\pi i}\int_{\beta^*-i\infty}^{\beta^*+i\infty}\frac{\Phi_0(u)}{c_0}\gamma_1(-u)\,du.$$

But

$$\frac{1}{2\pi i}\int_{\beta^*-i\infty}^{\beta^*+i\infty}\Phi_0(u)\gamma_1(-u)\,du = \int_0^\infty \varphi_0(x)g(x)\,dx\cdot c_0,$$

where $\varphi_0(x)$ is the solution of the homogeneous equation whose Laplace transform decreases as $1/\operatorname{Re}u$ as $\operatorname{Re}u\to\infty$. Consequently,

$$f(x) = \frac{1}{\sigma_+(-u_1)}\int_0^\infty \varphi_0(x)g(x)\,dx\cdot e^{-u_1x} + O(e^{-\sigma x}), \qquad u_1<\sigma<u^*.$$

When the characteristic equation (2) has a double root at the point $u=0$, then from the condition that the function $\Phi(u)$ have no pole of second order at $u=0$ we shall have:

$$f(x) = \frac{1}{\sigma_+(0)}\int_0^\infty g(x)\varphi_0(x)\,dx + O(e^{-\sigma x}).$$

And finally, in the case where the characteristic equation has two imaginary roots $i\nu$ and $-i\nu$, there exist two linearly independent solutions of the inhomogeneous equation, whose asymptotics are given by the expressions

$$f_1(x) = \frac{1}{\sigma_+(-i\nu)}\left[\int_0^\infty g(x)\varphi_0(x)\,dx\right]e^{-i\nu x} + O(e^{-\sigma x}),$$

$$f_2(x) = \frac{1}{\sigma_+(i\nu)}\left[\int_0^\infty g(x)\varphi_0(x)\,dx\right]e^{i\nu x} + O(e^{-\sigma x}).$$

If Fock’ s orthogonality condition

$$\int_0^\infty g(x)\varphi_0(x)\,dx = 0$$

is satisfied, then in all three cases we obtain a unique solution which, as $x \to \infty$, decreases as $e^{-\sigma x}$, $\sigma \sim u^*$.

The work was carried out under the supervision of Prof. E. S. Kuznetsov, to whom the author is indebted for constant attention and valuable advice.

Received
27 XII 1956

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