PHYSICS

1957

SovietRxiv

Abstract

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PHYSICS

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ON THE THEORY OF THE DECAY OF A QUASISTATIONARY STATE

(Presented by Academician V. A. Fock on 20 I 1957)

The paper considers certain questions in the theory of the decay of a quasistationary (almost stationary) state. The theory under consideration is of great importance in the study of α-decay, the passage of particles through potential barriers, the distribution of the energy levels of a nucleus, and other problems ([1,2]).

1. Let $\psi_0 = \psi(x, 0)$ be the wave function describing the state of a physical system at the time $t = 0$*. The probability that after a time t the system will still be in the initial state ψ_0 is determined by the formula

$$L(t) = |p(t)|^2, \tag{1}$$

where

$$p(t) = \int e^{-\frac{i}{\hbar}Et}\,\omega(E)\,dE; \tag{2}$$

$\omega(E)$ is the density of the energy distribution in the initial state (and, consequently, also for states at the time t). The theorem indicated above on the relation between the decay law of an almost stationary state and the energy-distribution function in this state was obtained by V. A. Fock and N. S. Krylov in their investigation of the uncertainty relation for energy and time ([1]). In the same paper it was shown that, in the problem of the escape of a particle from a potential well through a potential-energy barrier, the distribution density $\omega(E)$ is a meromorphic function of the complex variable E. Taking into account only the pair of poles of $\omega(E)$ nearest to the real axis,

$$E = E_0 \pm i\Gamma; \qquad E_0 > 0; \qquad \Gamma > 0, \tag{3}$$

for sufficiently large times t we obtain from formula (1)

$$L(t) = e^{-\frac{2\Gamma}{\hbar}t}, \tag{4}$$

which expresses the usual exponential law of decay of a quasistationary state. In this case $\omega(E)$ is determined by the expression

$$\omega(E) = \frac{1}{\pi} \frac{\Gamma}{(E - E_0)^2 + \Gamma^2}, \tag{5}$$

i.e., by the usual dispersion formula for the energy distribution.

2. For further investigations we shall refine certain basic formulas. In all formulas the integration over energy is carried out over the region of the continuous spectrum $E \in (0, \infty)$, so that

$$p(t) = \int_{\infty}^{\infty} e^{-\frac{i}{\hbar}Et}\, \omega(E)\, dE, \tag{6}$$

* We follow the notation of work ([1]).

where the introduced function $\overline{\omega}(E)$ is "semi-finite" :

$$\overline{\omega}(E) = \begin{cases} \omega(E), & E > 0; \\ 0, & E \leq 0. \end{cases} \tag{7}$$

From (6), obviously, we obtain:

$$\overline{\omega}(E) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}Et}\, p(t)\, dt, \tag{8}$$

so that knowledge of $p(t)$ makes it possible, on the basis of (8), to determine analytically the distribution function $\overline{\omega}(E)$.

The restriction (7) on the admissible distribution functions $\overline{\omega}(E)$ imposes certain restrictions on the functions $p(t)$. Namely, introducing

$$p_1(t) \equiv p(-t) = \int_{-\infty}^{\infty} e^{\frac{i}{\hbar}Et}\, \overline{\omega}(E)\, dE, \tag{9}$$

one can show that the integral (9) defines $p_1(t)$ as an analytic function in the upper half-plane $\operatorname{Im} t > 0$. Using Cauchy's formula

$$\frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{p_1(t')}{t' - t}\, dt' = p_1(t) + \sum_i \frac{1}{t_i - t} \operatorname{Res} p_1(t_i), \tag{10}$$

we obtain the desired "dispersion" relations, separating in (10) the real and imaginary parts. Assuming that $p_1(t)$ has no poles on the real axis and using, in addition, the equalities following from the reality of $\overline{\omega}(E)$, we obtain the following "dispersion" relations*:

$$\operatorname{Re} p(t) = -\frac{2}{\pi} P \int_0^\infty \frac{t' \operatorname{Im} p(t') - t \operatorname{Im} p(t)}{{t'}^2 - t^2}\, dt',$$

$$\operatorname{Im} p(t) = \frac{2t}{\pi} P \int_0^\infty \frac{\operatorname{Re} p(t') - \operatorname{Re} p(t)}{{t'}^2 - t^2}\, dt'. \tag{11}$$

Thus, the fact that the distribution function $\overline{\omega}(E)$ is "semi-finite" leads to the real and imaginary parts of the function $p(t)$ being connected with one another by integral "dispersion" relations.

Introducing the modulus $M(t)$ and phase $N(t)$ of the function $p(t)$:

$$p(t) = M(t) \exp(iN(t));$$

$$\operatorname{Re} p(t) = M(t) \cos N(t); \qquad \operatorname{Im} p(t) = M(t) \sin N(t), \tag{12}$$

it is easy to obtain the following "dispersion" relations:

$$\operatorname{Re} p(t) = M(t) \cos N(t) = -\frac{2}{\pi} P \int_0^\infty \frac{t' M(t') \sin N(t') - t M(t) \sin N(t)}{{t'}^2 - t^2}\, dt',$$

$$\operatorname{Im} p(t) = M(t) \sin N(t) = \frac{2t}{\pi} P \int_0^\infty \frac{M(t') \cos N(t') - M(t) \cos N(t)}{{t'}^2 - t^2}\, dt', \tag{13}$$

i.e., relations connecting the modulus $M(t)$ (the square of which is the physically measurable quantity $L(t)$) and the phase $N(t)$ of the function $p(t)$.

* In deriving (10), and consequently also (11), it is assumed that $p_1(t)$ decreases sufficiently rapidly as $|t| \to$

Analogous "dispersion" relations can be derived in the more general case when $\bar{\omega}(E)$ is "semifinite" in the sense:

$$\bar{\omega}(E) = \begin{cases} \omega(E), & E > E_1; \\ 0, & E \leqslant E_1. \end{cases} \tag{14}$$

3. Another consequence of the "semifiniteness" of $\bar{\omega}(E)$ is a criterion which must be satisfied by the modulus $M(t)$ of the function $p(t)$, which has direct physical meaning. The formulation of the problem is as follows: suppose a nonnegative function $M(t)$ is given; is it possible to choose such a real function $N(t)$ that the function $\bar{\omega}(E)$, whose Fourier transform is $p(t)$, is "semifinite" in the sense of (14).

It turns out that arbitrary $M(t)$ are inadmissible. The restrictions which admissible $M(t)$ must satisfy follow from the fundamental Paley–Wiener theorem ([3]). In the theory of the decay of a quasistationary state this theorem can be formulated as follows:

In order that $L(t)$ represent the probability of decay of a quasistationary state ψ_0, it is necessary that the following criterion be fulfilled:

$$\int_{-\infty}^{\infty} \frac{|\log M(t)|}{1+t^2}\, dt < \infty, \tag{15}$$

where $L(t) = M^2(t)$.

This criterion, which the modulus $M(t)$ of the function $p(t)$ must necessarily satisfy, we shall call the **criterion of finiteness** or **of physical realizability** ([4]) in the theory of the decay of a quasistationary state. It is necessary to note that whereas the "dispersion" relations depend essentially on the limit of "semifiniteness" E_1, the criterion of finiteness or physical realizability is independent of it altogether (it is only necessary that the limit E_1 be finite). Since the integral (15), if it converges, is absolutely convergent, the signs of its convergence are the Cauchy signs ([5]).

Criterion (15) allows the following remarks to be made concerning the function $M(t)$:

a) the function $M(t)$ cannot vanish on some interval of the variable t, and still less can it be

$$L(t) = 0 \quad \text{for } t \geqslant T_0; \tag{16}$$

b) as $|t| \to \infty$, convergence of the integral (15) is ensured by fulfillment of the inequality

$$\frac{|\log M(t)|}{1+t^2} < \frac{A}{t^l}, \tag{17}$$

where $A > 0,\ l > 1$;

c) convergence at special points t_0: $M(t_0) = 0,\ M(t_0) = \infty$ is ensured by fulfillment of the inequality

$$\frac{|\log M(t)|}{1+t^2} < \frac{A_1}{(t-t_0)^{l_1}}, \tag{18}$$

where $A_1 > 0,\ l_1 < 1$.

The inequalities obtained, (17), (18), are of fundamental importance in the theory of the decay of a quasistationary state, especially inequality (17). Indeed, inequality (17) means that as $|t| \to \infty$, in any case:

$$L(t) \geqslant Ae^{-\gamma|t|^q}, \tag{19}$$

where $A > 0$, $\gamma > 0$, $q < 1$, i.e., there can be no fundamentally exponential law of decay for all $t \in (0, \infty)$. In Ref. [1], however, an exponential law of decay was obtained because the calculations were carried out approximately. Taking into account that $\bar{\omega}(E)$ in (5) is equal to zero for $E \leq 0$, we obtain

$$p(t) = \frac{1}{\pi}\int_0^\infty e^{-\frac{i}{\hbar}Et}\,\frac{\Gamma}{(E-E_0)^2+\Gamma^2}\,dE \simeq e^{-\frac{i}{\hbar}E_0 t - \frac{\Gamma}{\hbar}t} - \frac{i}{\pi}\,\frac{\Gamma\hbar}{(E_0^2+\Gamma^2)t} \tag{20}$$

for

$$\frac{t}{\hbar}\sqrt{E_0^2+\Gamma^2} > 1.$$

The presence of the second, nonexponential term in (20) precisely confirms the criterion of physical realizability (15). It follows from (20) that the exponential law of decay is valid only in a finite range of values of $\Gamma t/\hbar$, namely for

$$\frac{\Gamma t}{\hbar}e^{-\Gamma t/\hbar} \gg \left(\frac{\Gamma}{E_0}\right)^2. \tag{21}$$

Similar results can also be obtained in the more general case where $\bar{\omega}(E)$ differs from expression (5) by a factor $\sqrt{E}$.

4. If $p_1(t)$ has no zeros in the half-plane $\operatorname{Im} t > 0$, then one can obtain a relation directly connecting $M(t)$ and $N(t)$:

$$N(t) = \frac{2t}{\pi}P\int_0^\infty \frac{\log M(t') - \log M(t)}{t'^2 - t^2}\,dt', \tag{22}$$

so that, knowing $M(t')$ (from an experimental determination of the decay law $L(t')$), one can determine, on the basis of (22), $N(t)$ uniquely, and consequently also $\bar{\omega}(E)$ on the basis of (8).

If, however, $p_1(t)$ has in the half-plane $\operatorname{Im} t > 0$, for example, a zero at $t_n = \alpha_n + i\beta_n$, $\beta_n > 0$, then (22) determines $N(t)$ up to

$$\operatorname{arc\,tg} \frac{2(t-\alpha_n)\beta_n}{(t-\alpha_n)^2 - \beta_n^2}. \tag{23}$$

In conclusion I express my gratitude to Academician V. A. Fock for discussion of the work and valuable comments, and also to the members of the seminar of the Department of Theoretical Physics of Leningrad State University for discussion of the work.

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of Exploration Geophysics

Received
23 I 1957

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Note: Figure translations are in progress. See original paper for figures.