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# MATHEMATICS

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## Abstract

## Full Text

MATHEMATICS

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# ON A DIFFERENCE METHOD FOR SOLVING A NONLINEAR GOURSAT PROBLEM

*(Presented by Academician S. L. Sobolev on 31 V 1957)*

Difference methods have been applied to the solution of the Goursat problem in the case of two independent variables in papers <sup>(1-3)</sup>. In <sup>(1, 2)</sup> the right-hand side of equation (1) was assumed not to depend on  $u_x, u_y$ , while in <sup>(3)</sup> it depended linearly on  $u_x, u_y$ .

In the present note we consider the solution by the method of finite differences of the Goursat problem

$$u_{xy} = f(x, y, u, u_x, u_y), \quad (1)$$

$$u(x, 0) = \varphi(x), \quad 0 \leq x \leq l_x; \quad u(0, y) = \psi(y), \quad 0 \leq y \leq l_y; \quad \varphi(0) = \psi(0) \quad (2)$$

for an arbitrary right-hand side  $f(x, y, u, u_x, u_y)$ , continuous in the totality of all its arguments in the domain of definition

$$D: \quad 0 \leq x \leq l_x, \quad 0 \leq y \leq l_y;$$

$$|u - u^0| \leq l_u, \quad |u_x - u_x^0| \leq l_{u_x}, \quad |u_y - u_y^0| \leq l_{u_y} \quad (3)$$

and sufficiently smooth with respect to  $u, u_x, u_y$ , for arbitrary continuously differentiable boundary functions  $\varphi(x), \psi(y)$ . In Sec. 1, on the basis of a certain general convergence criterion, the existence of a continuously differentiable solution is established in the case where  $f$  satisfies the Lipschitz condition with respect to  $u_x$  and  $u_y$ . In Sec. 2 the existence and uniqueness of the solution are established in the case where the Lipschitz condition is also satisfied with respect to  $u$ .\* In Sec. 3 an a priori error estimate is given in terms of the steps in  $x$  and  $y$  and the maxima of the moduli of the given functions, their first derivatives, and the Lipschitz constants, under the assumption that  $f, \varphi'(x), \psi'(y)$  satisfy the Lipschitz condition with respect to all their arguments. In Sec. 4 the results are extended to a system of equations in  $m$ -dimensional space.

1°. Consider a function of two points  $f(P, Q)$ , where  $P$  ranges over a closed bounded domain  $G$  of Euclidean space  $R$ , and  $Q$  over an arbitrary domain  $G^*$  of Euclidean space  $R^*$ . We shall regard the function  $f(P, Q)$  as a family  $\mathfrak{M}$  of functions defined in  $G$  and depending on the parameter  $Q$ .

A sequence of functions  $f(P, Q_n)$ , where  $Q_n \rightarrow Q$  as  $n \rightarrow +\infty$ , will be called **uniformly bounded** if there exists a constant  $C$  such that  $|f(P, Q_n)| < C$  for all  $n$  and at once for all  $P \in G$ .

A sequence of functions  $f(P, Q_n)$ , where  $Q_n \rightarrow Q$  as  $n \rightarrow +\infty$ , will be called **equally smoothing** if

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\* Under these conditions the existence is proved more simply.

for every  $\varepsilon > 0$  there exist such  $n_0 = n_0(\varepsilon)$  and  $\delta = \delta(\varepsilon)$  that the inequality

$$|f(P', Q_n) - f(P'', Q_n)| < \varepsilon$$

will hold for all  $n > n_0 = n_0(\varepsilon)$  and simultaneously for all  $P'$  and  $P''$  from  $G$  for which the distance  $\rho(P', P'') < \delta(\varepsilon)$ .

Obviously, a uniformly bounded and equicontinuously smoothing sequence may consist of discontinuous functions.

**Theorem 1.** *Let the point  $Q_0$  belong to the domain  $G^*$  or to its boundary. We shall consider all possible sequences of points  $\{Q_n\}$  from the domain  $G^*$  converging to the point  $Q_0$ , and the corresponding sequences of functions  $\{f(P, Q_n)\}$  belonging to the family  $\mathfrak{M}$ . In order that from each sequence  $\{f(P, Q_n)\}$  one can extract a subsequence converging uniformly in the domain  $G$  to a continuous function, it is necessary and sufficient that from each such sequence one can extract a subsequence that is uniformly bounded and equicontinuously smoothing.*

**Remark.** If  $\mathfrak{M} \subset C$ , where  $C$  is the space of continuous functions defined in the usual way, then this theorem becomes a criterion of compactness of  $\mathfrak{M}$  in  $C$ . However, in the present note we shall not use this.

Let us proceed to the construction of a solution of the Goursat boundary-value problem (1), (2), (3) by the finite-difference method. In order not to pass to the consideration of subdomains, we shall assume that

$$u^0 = u(0, 0) = \varphi(0) = \psi(0), \quad u_x^0 = u_x(0, 0) = \varphi'(0), \quad u_y^0 = u_y(0, 0) = \psi'(0).$$

We prescribe along the  $x$ -axis a step  $h$  and along the  $y$ -axis a step  $k$ , putting  $x_i = ih$ ,  $i = 0, 1, 2, \dots$ ;  $y_j = jk$ ,  $j = 0, 1, 2, \dots$ ;  $0 \leq x_i < l_x$ ,  $0 \leq y_j < l_y$ . We replace equation (1) by the difference equation

$$\Delta_{hk}^2 u_{ij} = hkf(x_i, y_j, u_{ij}, u_{xij}, u_{yij}), \quad (1_{\Delta})$$

where the indices  $i, j$  indicate the values of the quantities referred to the node with coordinates  $x_i, y_j$ , and the differences  $\Delta_h u_{ij}, \Delta_k u_{ij}, \Delta_{hk}^2 u_{ij}$  are defined in the usual manner:

$$\Delta_h u_{ij} = u_{i+1,j} - u_{ij}, \quad \Delta_k u_{ij} = u_{i,j+1} - u_{ij}, \quad (4)$$

$$\Delta_{hk}^2 u_{ij} = \Delta_k(\Delta_h u_{ij}) = u_{i+1,j+1} - u_{i+1,j} - u_{i,j+1} + u_{ij}.$$

Here  $u_{xij}$  and  $u_{yij}$  denote the difference quotients

$$u_{xij} = \frac{\Delta_h u_{ij}}{h}, \quad u_{yij} = \frac{\Delta_k u_{ij}}{k}. \quad (5)$$

We replace the boundary conditions (2) by the boundary conditions

$$u_{0i} = \varphi(x_i), \quad i = 0, 1, 2, \dots; \quad u_{0j} = \psi(y_j), \quad j = 0, 1, 2, \dots \quad (2_{\Delta})$$

The solution of the difference equation (1<sub>Δ</sub>) under the boundary conditions (2<sub>Δ</sub>) is feasible in the domain defined by the inequalities

$$0 \leq x_i \leq l_x, \quad 0 \leq y_j \leq l_y,$$

$$\max_{0 \leq x \leq l_x} |\varphi(x) - \varphi(0)| + \max_{0 \leq y \leq l_y} |\psi(y) - \psi(0)| + x_i y_j f_{\max} \leq l_u,$$

$$|\varphi'(0)| + \varphi'_{\max} + y_j f_{\max} \leq l_{u_x}, \quad |\psi'(0)| + \psi'_{\max} + x_i f_{\max} \leq l_{u_y}. \quad (6)$$

\* Recall that the domain  $G$  is assumed to be closed and bounded.

where by  $f_{\max}, \varphi'_{\max}, \psi'_{\max}$  here and below is denoted the maximum of the modulus of the quantities  $f(x, y, u, u_x, u_y), \varphi'(x)$ , and  $\psi'(y)$  in the domain of their definition. Let us construct an "approximate solution"  $\tilde{u}(x, y; h, k)$  of the problem (1), (2), (3), by setting

$$\begin{aligned} \tilde{u}(x, y; h, k) = & u_{ij} + \frac{\Delta_h u_{ij}}{h}(x - x_i) + \frac{\Delta_k u_{ij}}{k}(y - y_j) + \\ & + \frac{\Delta_{hk}^2 u_{ij}}{hk}(x - x_i)(y - y_j) \end{aligned} \quad (7)$$

in each cell

$$\Pi_{ij}: \quad x_i \leq x < x_{i+1}, \quad y_j \leq y < y_{j+1}. \quad (8)$$

It is continuous in the domain (6), and its partial derivatives

$$\tilde{u}_x(x, y; h, k), \quad \tilde{u}_y(x, y; h, k) \quad (9)$$

are continuous in each cell  $\Pi_{ij}$  and piecewise continuous in the domain (6).

**Theorem 2.** If  $f(x, y, u, u_x, u_y)$  is continuous jointly in all arguments in the domain (3) and satisfies the Lipschitz condition with respect to  $u_x, u_y$ , then the families of functions (7) and (9) are uniformly bounded and equicontinuous in the domain (6) as  $h$  and  $k$  tend to zero, and, consequently, by Theorem 1, one can find a sequence of pairs  $(h_n, k_n)$ ,  $h_n \rightarrow 0$ ,  $k_n \rightarrow 0$  as  $n \rightarrow +\infty$ , such that

$$\tilde{u}(x, y; h_n, k_n) \rightrightarrows u(x, y),$$

$$\tilde{u}_x(x, y; h_n, k_n) \rightrightarrows v(x, y), \quad \tilde{u}_y(x, y; h_n, k_n) \rightrightarrows w(x, y)$$

uniformly in the domain (6), where  $u, v, w$  are continuous in this domain.

**Theorem 3.** In the domain (6), under the assumptions of Theorem 2, the function  $u(x, y)$  has continuous first partial derivatives  $u_x, u_y$ , equal respectively to  $v, w$ , has a continuous partial derivative  $u_{xy}$ , and satisfies equation (1) and the boundary conditions (2).

2°. **Theorem 4.** If  $f(x, y, u, u_x, u_y)$  is continuous jointly in all arguments and satisfies the Lipschitz condition with respect to the last three arguments

$$\begin{aligned} & |f(x, y, \bar{u}, \bar{u}_x, \bar{u}_y) - f(x, y, \bar{\bar{u}}, \bar{\bar{u}}_x, \bar{\bar{u}}_y)| \leq \\ & \leq L\{|\bar{u} - \bar{\bar{u}}| + |\bar{u}_x - \bar{\bar{u}}_x| + |\bar{u}_y - \bar{\bar{u}}_y|\}, \end{aligned} \quad (10)$$

then for the families (7) and (9), as  $(h^2 + k^2) \rightarrow 0$ , the Cauchy criterion for uniform convergence is fulfilled. For their limits  $u(x, y), v(x, y), w(x, y)$ , Theorem 3 holds. The boundary-value problem (1), (2), (3) cannot have two different continuously differentiable solutions in the domain (6).

3°. **Theorem 5.** If the function  $f(x, y, u, u_x, u_y)$  satisfies the Lipschitz condition with constant  $L$  with respect to all arguments, and  $\varphi'(x), \psi'(y)$  satisfy the Lipschitz condition with constants  $L_{\varphi'}, L_{\psi'}$ , then the following a priori error estimate holds:

$$\begin{aligned}
 & |\tilde{u} - u| + |\tilde{u}_x - u_x| + |\tilde{u}_y - u_y| \leq \\
 & \leq \varepsilon + \varepsilon L \left\{ \frac{2}{LM} [e^{LM(x+y)} - 1] + \frac{1}{L^2 M^2} [e^{LM(x+y)} - 1 - LM(x+y)] \right\} \quad (11)
 \end{aligned}$$

in the domain (6), where

$$M = 2 + l_x + l_y, \quad (12)$$

$$\begin{aligned}
 \varepsilon = \varepsilon(h, k) = & 4(\varphi'_{\max} h + \psi'_{\max} k) + 2(L_\varphi h + L_\psi k) + \\
 & + 2L\{l_x + l_y + l_x l_y\} \{ [1 + \varphi'_{\max} + (1 + l_y) f_{\max}] h + \\
 & + [1 + \psi'_{\max} + (1 + l_x) f_{\max}] k + h k f_{\max} \}; \quad (13)
 \end{aligned}$$

and  $u$  is the exact solution of problem (1), (2), (3);  $\tilde{u}$  is the approximate solution of the problem, defined in § 1.

4°. Consider the boundary-value problem

$$u_{x_1 x_2 \dots x_m}^{(\nu)} = f_\nu(x_1, x_2, \dots, x_m; u^{(1)}, \dots, u^{(N)}, u_{x_1}^{(1)}, \dots, u_{x_2 \dots x_m}^{(N)}), \quad 1 \leq \nu \leq N; \quad (14)$$

$$u^{(\nu)}|_{x_i=0} = \varphi_i^{(\nu)}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m), \quad 1 \leq \nu \leq N, \quad 1 \leq i \leq m, \quad (15)$$

where  $\varphi_i^{(\nu)}$  and  $f_\nu$  are defined and continuous for  $0 \leq x_j \leq l_{x_j}$ ,  $j = 1, \dots, m$ ,

$$|u^{(\nu)} - u^{(\nu)0}| \leq l_{u^{(\nu)}}, \dots, |u_{x_1 \dots x_m}^{(\nu)} - u_{x_1 \dots x_m}^{(\nu)0}| \leq l_{u_{x_1 \dots x_m}^{(\nu)}}, \quad 1 \leq \nu \leq N,$$

and in  $f_\nu$  there enter only the mixed derivatives of the functions  $u^{(\nu)}$  up to order  $\leq m - 1$ , while the boundary functions  $\varphi_i^{(\nu)}$  have all possible mixed continuous derivatives up to order  $m - 1$  inclusive and satisfy the natural compatibility conditions.

By a solution of problem (14), (15) we shall mean a system of functions  $u^{(\nu)}$  satisfying (14), (15) and continuous together with the mixed derivatives up to order  $m$  inclusive.

Theorems 2, 3, 4, and 5 extend to the boundary-value problem (14), (15). In particular, if  $f_\nu$  satisfies a Lipschitz condition in all arguments with constant

$L$ , and the  $(m-1)$ -st derivatives of  $\varphi_i^{(\nu)}$  also satisfy a Lipschitz condition in all arguments with constants, respectively,  $L_i^{(\nu)}$ , then for the error incurred in solving problem (14), (15) by the finite-difference method according to the scheme described above, we obtain the estimate

$$\begin{aligned} & \sum_{\nu=1}^N \left\{ |\tilde{u}^{(\nu)} - u^{(\nu)}| + \sum_{i_1=1}^m |\tilde{u}_{x_{i_1}}^{(\nu)} - u_{x_{i_1}}^{(\nu)}| + \dots \right. \\ & \left. \dots + \sum_{i_1, \dots, i_{m-1}=1}^m |\tilde{u}_{x_{i_1} \dots x_{i_{m-1}}}^{(\nu)} - u_{x_{i_1} \dots x_{i_{m-1}}}^{(\nu)}| \right\} \leq \\ & \leq \varepsilon + \varepsilon LM \left\{ \frac{C_m^{m-1}}{LNM} [e^\omega - 1] + \frac{C_m^{m-2}}{L^2 N^2 M^2} [e^\omega - 1 - \omega] + \dots \right. \\ & \left. \dots + \frac{C_m^{m-m}}{L^m N^m M^m} \left[ e^\omega - 1 - \omega - \dots - \frac{\omega^{m-1}}{(m-1)!} \right] + [(m-1)! - 1] \frac{\omega^m}{m!} \right\}, \end{aligned}$$

where  $\varepsilon = \varepsilon(h_1, h_2, \dots, h_m)$  is expressed analogously to (13) in terms of the steps  $h_1, h_2, \dots, h_m$  with respect to  $x_1, x_2, \dots, x_m$ ;  $C_m^k$  are binomial coefficients; the constant  $M$  is expressed in terms of the dimensions of the domain;  $\omega = LNM(x_1 + x_2 + \dots + x_m)$ .

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*

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