MATHEMATICS

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Abstract

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ON THE CAUCHY PROBLEM FOR A PARABOLIC EQUATION OF SECOND ORDER WITH VARIABLE COEFFICIENTS

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Consider the equation

$$\frac{\partial u}{\partial t} + Lu = f(x,t), \tag{1}$$

where

$$L = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x), \quad x = (x_1, \dots, x_n).$$

Introduce the notation:

$$\Delta(x) = \det \|a_{ij}(x)\|, \quad \Delta_i(x) = \begin{vmatrix} a_{11} \dots a_{1,i-1} & b_1 & a_{1,i+1} \dots a_{1n} \\ \dots\dots\dots\dots & & \\ \dots\dots\dots\dots & & \\ a_{n1} \dots a_{n,i-1} & b_n & a_{n,i+1} \dots a_{nn} \end{vmatrix}.$$

Suppose that the coefficients of the operator L satisfy the following conditions:

$$a_{ij}(x) = a_{ji}(x) \tag{2}$$

for all x,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha^2 \sum_{i=1}^{n} \xi_i^2 \quad (\alpha > 0) \tag{3}$$

and the coefficient $c(x)$ is bounded from below:

$$c(x) > -c \quad (c > 0). \tag{4}$$

Let there exist a differentiable function $g(x)$ such that

$$\frac{\partial g}{\partial x_i} = \frac{\Delta_i(x)}{\Delta(x)}. \tag{5}$$

To equation (1) we append the initial condition

$$u(x, 0) = 0. \tag{1'}$$

In the present paper we study questions of existence and uniqueness of the solution of the Cauchy problem (1)–(1′) in the whole space $(x_1, \dots, x_n)$ in the class of rapidly increasing functions.

Following M. I. Vishik ([1]), we define a generalized solution of the Cauchy problem (1)–(1′). For this we shall need to introduce into consideration certain Hilbert spaces of functions.

Let us first consider the space Ω_L of finite twice differentiable functions with the scalar product

$$(u,\ v) = \int_{-\infty}^{\infty} \cdots \int u(x,\ t) v(x,\ t) e^{-g(x)} dx. \tag{6}$$

We shall show that the operator L is symmetric on Ω_L. Let Ω be a domain in the space $(x_1, \dots, x_n)$; let Γ be its boundary; and let $\mathbf{n}$ be the outward normal to Γ. Putting, in the well-known Ostrogradsky formula,

$$\int_\Omega \cdots \int \operatorname{div} \mathbf{R} \, dx = \int_\Gamma \cdots \int (\mathbf{R}, \mathbf{n}) \, d\sigma$$

($(\mathbf{R}, \mathbf{n})$ is the ordinary scalar product of two vectors)

$$\mathbf{R} = \left\{ v(x) e^{-g(x)} \sum_{j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} \right\},$$

and taking for Ω a sphere of radius r, then letting r tend to infinity, we easily obtain the formula

$$-\int_{-\infty}^{\infty} \cdots \int \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} v(x) e^{-g(x)} dx = \tag{7}$$

$$= \int_{-\infty}^{\infty} \cdots \int \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} e^{-g(x)} dx - \int_{-\infty}^{\infty} \cdots \int \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} v(x) e^{-g(x)} \frac{\Delta_i(x)}{\Delta(x)} dx.$$

From (7) it follows that

$$(Lu,\ v) = \int_{-\infty}^{\infty} \cdots \int \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} e^{-g(x)} dx +$$

$$+ \sum_{j=1}^{n} \int_{-\infty}^{\infty} \cdots \int \left(-\sum_{i=1}^{n} a_{ij}(x) \frac{\Delta_i(x)}{\Delta(x)} + b_j(x) \right) \frac{\partial u}{\partial x_j} v(x) e^{-g(x)} dx +$$

$$+ \int_{-\infty}^{\infty} \cdots \int c(x) u(x) v(x) e^{-g(x)} dx.$$

But, by virtue of the definition of the quantities $\Delta_i(x)$ and $\Delta(x)$, it is obvious that

$$\sum_{i=1}^{n} a_{ij}(x) \frac{\Delta_i(x)}{\Delta(x)} = b_j(x) \quad (j = 1, \dots, n).$$

Therefore

$$(Lu,\ v) = \int_{-\infty}^{\infty} \cdots \int \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i} e^{-g(x)} dx + \int_{-\infty}^{\infty} \cdots \int c(x) u(x) v(x) e^{-g(x)} dx. \tag{8}$$

Interchanging in (8) the roles of the functions $u(x)$ and $v(x)$, and taking (2) into account, we obtain the symmetry of the operator L on Ω_L:

$$(Lu,\ v) = (u,\ Lv). \tag{9}$$

From formula (8) and conditions (3), (4) it follows that $(Lu, u) > -c(u, u)$. Making in (1) the substitution of the unknown function $u(x,t) = e^{k_1 t} v(x,t)$ and choosing k_1 so that $k = k_1 - c > 0$, we obtain that the operator $L_1 = L + k$ satisfies the condition

$$(L_1 u,\ u) > k(u,\ u). \tag{10}$$

Therefore one may assume that L also satisfies condition (10).

We now introduce in the space Ω_L a new metric by the formula

$$\{u, u\}' = (Lu, u). \tag{11}$$

Complete the space Ω_L with respect to the metrics (6) and (11). The resulting complete Hilbert spaces will be denoted respectively by H and H'.

Since the bilinear form (Lu, v) is bounded in the metric (11), it can be realized as the bilinear form of some bounded symmetric operator $\widetilde{L}$ in H'

$$(Lu, v) = \{\widetilde{L}u, v\}'. \tag{12}$$

We now consider the operator $S = \dfrac{\partial}{\partial t} + L$, defined on the space of functions Ω_S satisfying the following conditions:

1) $u(x, 0) = 0$;
2) $u(x, t) \in \Omega_L$ for all $t \in [0, T]$;
3) $\dfrac{\partial u}{\partial t}$ exists in H for $t \in [0, T]$, and moreover

$$\int_0^T \left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right) dt < \infty;$$

4)

$$\int_0^T (Lu, Lu)\, dt < \infty.$$

In the space of functions Ω_S we introduce two metrics by the formulas

$$[u, v] = \int_0^T (u, v)\, dt; \tag{13}$$

$$\{u, v\}_1 = \int_0^T \{u, v\}'\, dt. \tag{14}$$

Complete the space Ω_S with respect to the metrics (13) and (14). The resulting complete Hilbert spaces will be denoted respectively by $\mathfrak{A}$ and $\mathfrak{A}'$. Since

$$\left|\left[\frac{\partial u}{\partial t}, v\right]\right| \le \left[\frac{\partial u}{\partial t}, \frac{\partial u}{\partial t}\right] \cdot [v, v] \le C_1 \int_0^T (Lv, v)\, dt \le C_2 \{v, v\}_1,$$

the bilinear form $\left[\dfrac{\partial u}{\partial t}, v\right]$ can be realized in $\mathfrak{A}'$ in the form of the form of some operator $\widetilde{T}$

$$\left[\frac{\partial u}{\partial t}, v\right] = \{\widetilde{T}u, v\}_1.$$

At the same time

$$[Lu, v] = \int_0^T (Lu, v)\, dt = \int_0^T \{\widetilde{L}u, v\}'\, dt = \{\widetilde{L}u, v\}_1.$$

Thus,

$$[Su, v] = \{\widetilde{T}u, v\}_1 + \{\widetilde{L}u, v\}_1. \tag{15}$$

Now assuming that $v(x,t)$ belongs to the space Ω^*, consisting of functions $v(x,t)$ belonging to Ω_S and satisfying the condition $v(x,T) = 0$ (obviously, Ω^* is dense in Ω_S in the metric (14)), one may in (15) transfer the operator $\widetilde{T}$ to the function $v(x,t)$

$$[Su, v] = \{u, \widetilde{T}^* v\}_1 + \{u, \widetilde{L}^* v\}_1. \tag{16}$$

From (16) it follows that, if $u(x,t)$ is an ordinary solution of equation (1), then

$$[f, v] = \{u, (\widetilde{T}^* + \widetilde{L}^*) v\}_1. \tag{17}$$

By a generalized solution of problem (1)–(1′) we shall mean a function $u(x,t)$ belonging to the space $\mathfrak{A}'$, for which equality (17) holds for all $v(x,t)$ from the space Ω^* ([1]).

Since the operator L satisfies conditions (9) and (10) (symmetry and positive definiteness), it follows, by the results of ([1]) (see also ([2])), that the following theorem holds.

Theorem. *For any function $f(x,t)$ satisfying the condition*

$$\int_0^T (f, f)\, dt < \infty,$$

there exists a generalized solution of problem (1)–(1′), and it is unique.

Let us consider, as an example, an equation of the form (1), in which

$$Lu = -\Delta u + \sum_{i=1}^{n} b_i(x_i) \frac{\partial u}{\partial x_i} + c(x) u,$$

where Δ is the n-dimensional Laplace operator; b_i depends only on x_i; $x = (x_1, \dots, x_n)$. It is easy to see that conditions (2)–(5) are satisfied here.

In this case

$$g(x) = \sum_{i=1}^{n} \int_0^{x_i} b_i(y)\, dy.$$

Then for any function $f(x,t)$ satisfying the condition

$$\int_0^T \int_{-\infty}^{\infty} \cdots \int f^2(x,t) e^{-g(x)}\, dx\, dt < \infty,$$

there exists a generalized solution of the Cauchy problem (1)–(1′), and it is unique.

In particular, for $n = 1$, $b(x) = x^{2m-1}$, we obtain uniqueness of the solution of the Cauchy problem (1)–(1′) in the class of functions $u(x,t)$ satisfying the conditions

$$|u(x,t)| \le A_1 e^{A_2 |x|^{2m-\varepsilon}}, \qquad \left| \frac{\partial u(x,t)}{\partial x} \right| \le A_3 e^{A_4 |x|^{2m-\varepsilon}},$$

for any $\varepsilon > 0$; $0 \le t \le T$; A_1, A_2, A_3, A_4 are constants.

These examples show that the uniqueness class for the solution of the Cauchy problem (1)–(1′) can be extended without bound (in the sense of the admissible growth as $|x| \to \infty$ of the functions entering it), depending on the behavior as $|x| \to \infty$ of the coefficients of equation (1).

The mixed problem for equation (1) for the half-space $x_i > 0$ is posed as follows: one seeks a solution of equation (1) satisfying condition (1′) and the boundary condition

$$u(x_1, \dots, x_n, t)\Big|_{x_i=0} = 0. \qquad (1”)$$

Problem (1)–(1′)–(1″) is reduced in the usual way ([1]) to problem (1)–(1′). For this it is necessary to impose on the functions of the space Ω_L the additional restriction $u(x,t) = 0$ for $x_i \le 0$. Then, as above:

Theorem. *For any function $f(x,t)$ satisfying the condition*

$$\int_0^T (f, f)\, dt < \infty,$$

there exists a generalized solution of problem (1)–(1′)–(1″), and it is unique.

The examples analyzed above show that in this case as well the uniqueness class for the solution of problem (1)–(1′)–(1″) contains functions whose admissible growth as $|x| \to \infty$ can increase without bound as the growth as $|x| \to \infty$ of the coefficients of equation (1) increases.

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Note: Figure translations are in progress. See original paper for figures.