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Abstract

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THE PRINCIPLE OF MAJORANTS AND THE METHOD OF TANGENT PARABOLAS FOR NONLINEAR FUNCTIONAL EQUATIONS

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Suppose it is required to find an approximate solution of the functional equation

$$P(x) = 0, \tag{1}$$

where $y = P(x)$ is a nonlinear operation transforming $x \in X$ into $y \in Y$. By X and Y we mean complete linear spaces, normed in the sense of L. V. Kantorovich by means of linear semi-ordered spaces Z and W, spaces of type B_k ([1,2]).

We shall solve equation (1) by the method of successive approximations, in which the iterations are defined by the following equality ([3,4]):

$$x_{n+1} = x_n + \sum_{k=1}^{m} \frac{(-1)^k}{k!} \Phi^{(k)}(P(x_n))(P(x_n))^k, \tag{2}$$

where

$$\Phi'(P(x_n)) = [P'(x_n)]^{-1} = \Gamma_n, \qquad \Phi''(P(x_n)) = -\frac{1}{2}\Gamma_n P''(x_n)\Gamma_n^2$$

and so on.

Method (2) in the case $m = 1$ gives Newton' s method

$$x_{n+1} = x_n - \Gamma_{nP}(x_n),$$

considered in ([1]) for spaces of type B_k.

For $m = 2$ we obtain Chebyshev' s method

$$x_{n+1} = x_n - \Gamma_{nP}(x_n) - \frac{1}{2}\Gamma''_{nP}(x_n)[\Gamma_{nP}(x_n)]^2,$$

considered in ([4,5]) for Banach spaces and in ([6]) for spaces of type B_k.

It is natural to call process (2) an analogue of the process of tangent parabolas, since in the case of a real equation the finding of the approximation x_{n+1} from x_n is equivalent to finding the point of intersection of the axis of abscissas with the parabola

$$x = \sum_{k=0}^{m} a_{ky}^{k},$$

with appropriately chosen coefficients a_k, having with the curve $y = P(x)$ at the point $(x_n, P(x_n))$ contact of order m.

We shall consider a process analogous to (2) for the equation

$$Q(z) = 0, \tag{3}$$

where Q is an operation from Z to W.

Theorem 1. If in equations (1) and (3) P, Q are continuous operations having, respectively on the paths $x_0, x_1, x_2, \dots$ and $z_0, z_1, z_2, \dots$, weak derivatives in the sense of Gâteaux up to order $m+1$ inclusive, and the following conditions are fulfilled:

1) there exist $[P'(x_0)]^{-1} = \Gamma_0$ and $[Q'(z_0)]^{-1} = \Delta_0$, where Δ_0 is (oo)-continuous and $|\Gamma_0| \le -\Delta_0$;
2) $|\Gamma_0 P(x_0)| \le -\Delta_0 Q(z_0)$;
3) $|\Gamma_0 P^{(k)}(x_0)| \le -\Delta_0 Q^{(k)}(z_0)$ for $k = 2, 3, \dots, m;\ m > 1$ (for $m = 1$, condition 3) is omitted);
4) $|\Gamma_0 P^{(m+1)}(x)| \le -\Delta_0 Q^{(m+1)}(z)$ for $x \leftrightarrow z$ (to the point $x = x_n + \theta(x_{n+1} - x_n)$ there is assigned the point $z = z_n + \theta(z_{n+1} - z_n);\ 0 \le \theta \le 1$), then from the existence and convergence of the process of tangent parabolas for equation (3) there follows its existence and convergence for equation (1).

Proof. Denote by $\Delta x_{l,n}$ the general term of the sum (2)

$$\Delta x_{l,n} = \frac{(-1)^l}{l!} \Phi^{(l)}(P(x_n))(P(x_n))^l.$$

Then, according to (7), we shall have

$$\Delta x_{1,n} = -\Gamma_n P(x_n);$$

$$\Delta x_{l,n} = -\Gamma_n \sum_{k=2}^{l} \frac{1}{k!} P^{(k)}(x_n) \sum_{\substack{i_1+\cdots+i_{l-1}=k \\ 1\cdot i_1+\cdots+(l-1)i_{l-1}=l}} \frac{k!}{i_1!\cdots i_{l-1}!} \Delta x_{1,n}^{i_1} \cdots \Delta x_{l-1,n}^{i_{l-1}}. \quad (4)$$

From the condition of the theorem it follows directly that $|x_1 - x_0| \le z_1 - z_0$. Further, on the basis of Taylor' s formula,

$$\Gamma_0 P(x_1) = \Gamma_0 \sum_{k=0}^{m} \frac{P^{(k)}(x_0)}{k!}(x_1 - x_0)^k + \frac{1}{m!}\Gamma_0 \int_{x_0}^{x_1} P^{(m+1)}(\bar{x})(x_1 - \bar{x})^m \, d\bar{x}. \quad (5)$$

Putting $n = 0$ in (4) and substituting in (5), we obtain

$$|\Gamma_0 P(x_1)| = \left| \Gamma_0 \sum_{k=2}^{m} \frac{P^{(k)}(x_0)}{k!} \sum_{\substack{i_1+\cdots+i_m=k \\ 1\cdot i_1+\cdots+mi_m>m}} \frac{k!}{i_1!\cdots i_m!} \Delta x_{1,0}^{i_1} \cdots \Delta x_{m,0}^{i_m} + \frac{1}{m!}\Gamma_0 \int_{x_0}^{x_1} P^{(m+1)}(\bar{x})(x_1 - \bar{x})^m \, d\bar{x} \right| \le -\Delta \quad (6)$$

Since

$$|I - \Gamma_0 P'(x_1)| = \left| -\Gamma_0 \sum_{k=2}^{m} \frac{P^{(k)}(x_0)}{k!}(x_1 - x_0)^k - \frac{1}{m!}\Gamma_0 \int_{x_0}^{x_1} P^{(m+1)}(\bar{x})(x_1 - \bar{x})^m \, d\bar{x} \right| \le$$

$$\le I - \Delta_0 Q'(z_1),$$

we conclude analogously to (1) that there exists an operator $\Gamma_1 = [P'(x_1)]^{-1}$, and in this case

$$|\Gamma_1| \le -\Delta_1 = -[Q'(z_1)]^{-1}; \qquad |\Gamma_1 P(x_1)| \le -\Delta_1 Q(z_1).$$

It is now easy to verify that if in the conditions of the theorem the points x_0 and z_0 are replaced by x_1 and z_1, then these conditions will still be satisfied, and therefore the analogous process can be continued further.

Let us now consider the case when X and Y are Banach spaces.

Theorem 2. If

1) there exists $\Gamma_0 = [P'(x_0)]^{-1}$ and $\|\Gamma_0\| \le B$;
2) $\|\Gamma_0 P(x_0)\| \le \eta$;

3) $\|P^{(k)}(x_0)\| \leq M_k$ for $k = 2, 3, \dots, m$; $m > 1$ (for $m = 1$, condition 3) is omitted);
4) $\|P^{(m+1)}(x)\| \leq M_{m+1}$ in a domain containing the path $x_0, x_1, x_2, \dots$ (it is sufficient that $\|x - x_0\| \leq z^* \leq 2\eta$) and the condition is satisfied

$$h = B\eta \sum_{k=2}^{m+1} \frac{M_k}{(k-2)!} (2\eta)^{k-2} \leq \frac{1}{2} \tag{7}$$

for the existence of a positive root z^* (where z^* is the least positive root) of the equation

$$Q(z) = \frac{\eta}{B} - \frac{z}{B} + \sum_{k=2}^{m+1} \frac{M_k z^k}{k!} = 0, \tag{8}$$

then the process of tangent parabolas converges to the solution x^* of equation (1), which exists and lies in the domain

$$\|x - x_0\| \leq z^*. \tag{9}$$

The rate of convergence is characterized by the inequality

$$\|x^* - x_n\| \leq z^* - z_n. \tag{10}$$

Remark. It should be noted that the estimates (9) and (10), obtained in Theorem 2, cannot be improved, since they are attained for equation (8).

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