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Abstract

Full Text

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ON FROMMER' S METHOD FOR INVESTIGATING A SINGULAR POINT

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Let us consider the differential equation

$$dy/dx = [Y_n(x,y) + Y(x,y)]/[X_n(x,y) + X(x,y)]. \quad (1)$$

Here $X_n(x,y)$ and $Y_n(x,y)$ are homogeneous polynomials of degree n; X and Y are analytic functions containing terms of order higher than n. If the equation of possible tangents $xY_n(x,y) - yX_n(x,y) = 0$ has real roots, then the question of whether there exist characteristics entering the origin with a given tangent or not, and, if such characteristics exist, whether their set is finite or infinite, is very complicated and in the general case has not been completely solved. The most effective method for this investigation appears to us to be M. Frommer's method ([1]), based on the study of the order and measure of curvature of characteristics entering the origin; unfortunately, he outlined it only in the most general terms. The present work is devoted to the further development of this method.

Let us give the definition of the order and measure of curvature. Let L be some segment of a characteristic of equation (1) in the interval $(0, x)$ for $x > 0$. Let x and y be the current coordinates on L; λ a real parameter. It has been proved that the function $u(x,\lambda) = y/x^\lambda$ tends to a definite limit (finite or infinite) as x tends to zero from the right; denote this limit by $k(\lambda)$. Consider two sets: the set E_1 of values of λ for which $k(\lambda) = 0$, and the set E_2 of values of λ for which $k(\lambda) = \infty$.

The following possibilities may occur:

1. The set E_2 is empty. In this case it is easy to show that all λ belong to the set E_1; the characteristic L has infinite order of curvature.
2. The set E_1 is empty. All λ belong to the set E_2, and the characteristic L has infinite negative order of curvature.
3. The sets E_1 and E_2 are nonempty. In this case the set E_1 is bounded above, and the set E_2 is bounded below. The exact upper bound $\bar{\lambda}$ of the set E_1 (it coincides with the exact lower bound of the set E_2) we shall call the **order of curvature of the characteristic** L, and the quantity $k(\bar{\lambda}) = \gamma$ the **measure of curvature of the characteristic** L. The order of curvature $\bar{\lambda}$ belongs to the set E_1, or to E_2, or belongs to neither

of these sets, depending on whether the measure of curvature γ is equal to zero, or infinity, or is distinct from both zero and infinity. If the order of curvature $\bar{\lambda}$ is equal to zero, then the characteristic L enters the origin only on condition that the measure of curvature is also equal to zero. In general, if $\bar{\lambda}$ is the order of curvature, then $k(\bar{\lambda}-\varepsilon)=0$ and $k(\bar{\lambda}+\varepsilon)=\infty$ for every $\varepsilon>0$. This means that the sets E_1 and E_2 may be regarded respectively as the lower and upper Dedekind classes of the cut defining the order of curvature $\bar{\lambda}$. To find which orders and measures of curvature are possible, we write equation (1) in the form

$$\frac{dy}{dx}=\frac{a_0(x)+ya_1(x)+\cdots+y^{n-1}a_{n-1}(x)+y^n a_n(x)+y^{n+1}F_1(x,y)}{b_0(x)+yb_1(x)+\cdots+y^{p-1}b_{p-1}(x)+y^p b_p(x)+y^{p+1}F_2(x,y)}, \quad (2)$$

$$\begin{aligned}
a_0(x)&=x^{k_0}(a_{0,0}+a_{0,1}x+a_{0,2}x^2+\cdots),\ldots,\\
a_{n-1}(x)&=x^{k_{n-1}}(a_{n-1,0}+a_{n-1,1}x+a_{n-1,2}x^2+\cdots),\\
a_n(x)&=a+a_{n,1}x+a_{n,2}x^2+\cdots;\\
b_0(x)&=x^{m_0}(b_{0,0}+b_{0,1}x+b_{0,2}x^2+\cdots),\ldots,\\
b_{p-1}(x)&=x^{m_{p-1}}(b_{p-1,0}+b_{p-1,1}x+b_{p-1,2}x^2+\cdots),\\
b_p(x)&=b+b_{p,1}x+b_{p,2}x^2+\cdots.
\end{aligned}$$

The exponents $k_0,\ldots,k_n,\ m_0,\ldots,m_p$ are positive, and among the coefficients a and b at least one is different from zero (otherwise the right-hand side of equation (2) is reducible by x). The functions $F_1(x,y)$ and $F_2(x,y)$ are analytic.

Introducing into equation (2) the substitution $y=ux^\lambda$, we obtain

$$du/dx=[f_1(x,u)-\lambda u f_2(x,u)]/xf_2(x,u) \quad (3)$$

where

$$\begin{aligned}
f_1(x,u)&=x^{-\lambda}a_0(x)+ua_1(x)+u^2x^\lambda a_2(x)+\cdots+u^n x^{(n-1)\lambda}a_n(x)\\
&\quad+u^{n+1}x^{n\lambda}\theta_1(x,u),\\
f_2(x,u)&=x^{-1}b_0(x)+ux^{\lambda-1}b_1(x)+u^2x^{2\lambda-1}b_2(x)+\cdots\\
&\quad\cdots+u^p x^{p\lambda-1}b_p(x)+u^{p+1}x^{(p+1)\lambda-1}\theta_2(x,u);
\end{aligned}$$

the functions $\theta_1(x,u)$ and $\theta_2(x,u)$ are continuous in a neighborhood of the axis $x=0$.

The numerator and denominator of the right-hand side of equation (3) contain various powers of x, depending on the parameter λ. To clarify which of these powers is the smallest, we construct the following scheme (see Fig. 1): on the Ox axis we plot the parameter λ, which we take as the independent variable, and on the ordinate axis—the exponents of x in the expressions $f_1(x,u)$ and $f_2(x,u)$ which, for certain values of λ, may be the smallest, namely:

$$e_1 = (n-1)\lambda, \quad e_2 = (n-2)\lambda + k_{n-1}, \dots, e_n = k_1, \quad e_{n+1} = k_0 - \lambda;$$
$$\bar{e}_1 = p\lambda - 1, \quad \bar{e}_2 = (p-1)\lambda - 1 + m_{p-1}, \dots, \bar{e}_p = \lambda + m_1 - 1, \qquad (4)$$
$$\bar{e}_{p+1} = m_0 - 1.$$

The remaining exponents of x in the expressions $f_1(x,u)$ and $f_2(x,u)$ cannot be the smallest for any $\lambda > 0$, and we do not plot them on the scheme. Thus, in our scheme we obtain a finite number of straight lines (4) (solid lines) and a finite number of straight lines (4′) (dashed).

Consider the broken line $ABCDEF$, composed of the straight lines of our scheme in such a way that between this broken line and the abscissa axis there are no other straight lines of the scheme. We call this broken line the **characteristic** one, and the abscissae of its vertices the **characteristic numbers**. For each value of λ, the smallest exponent of x in the numerator of the right-hand side of (3) is equal to the ordinate of the characteristic broken line at the corresponding point.

We have proved the following propositions:

1. For the existence of characteristics of equation (1), distinct from the axis Ox and entering the origin with infinite order of curvature, the following conditions are necessary and sufficient: a) the solid straight line $e_{n+1} = k_0 - \lambda$ is absent from the scheme ($a_0(x) \equiv 0$, i.e. the axis Ox is a characteristic); b) the solid straight line $e_n = k_1$ is located below the dashed straight line $\bar{e}_p = m_0 - 1$ ($k_1 < m_0 - 1$); c) if the sum of the exponents $k_1 + m_0$ is odd, then the inequality $a_{1,0}b_{0,0} > 0$ holds.
2. For the existence of characteristics of equation (1) entering the origin with zero order of curvature and distinct from the axis Oy, the following conditions are necessary and sufficient: a) the straight line $\bar{e}_1 = p\lambda - 1$ is absent from the scheme (i.e. the axis Oy is a characteristic); b) the extreme left link of the characteristic broken line is the dashed straight line $\bar{e}_i = (p-i+1)\lambda$, located below the solid straight line $e_1 = (n-1)\lambda$ (this occurs if at least one of the functions $b_i(x)$ contains a linear term of the form $b_{i,0}x$, with $i < n-1$); c) if the sum of the exponents $i+n$ is odd, then the inequality $ab_{i,0} > 0$ holds.

We shall distinguish three types of finite orders, distinct from zero, of

curvature: ordinary orders of curvature, quasi-special and special. **Ordinary orders of curvature** we shall call those orders to each of which there corresponds only a finite number of distinct measures of curvature, and these measures themselves are finite and different from zero.

Fig. 1

Figure 1: Fig. 1

To a **left quasi-special order of curvature** there corresponds only the zero measure of curvature, and to a **right quasi-special order** there corresponds only an infinite measure of curvature; moreover, in both cases the number of characteristics entering the origin is infinitely large. To a special order of curvature there corresponds an infinite set of distinct measures of curvature (including zero and infinity), and for any preassigned measure of curvature (except, possibly, a finite number of these measures) there exists one and only one characteristic entering the origin with this measure. We shall distinguish ordinary and special links of the characteristic polygonal line. An **ordinary link** is one that consists either only of a solid or only of a dotted straight line. A **special link** consists of two coincident straight lines: a solid and a dotted one. We have proved the following propositions:

Fig. 1

3. *Finite and nonzero orders of curvature corresponding to ordinary links of the characteristic polygonal line can only be ordinary. The ordinary orders of curvature are equal to the characteristic numbers.*
4. *A left quasi-special order of curvature can be equal only to the abscissa of the left end, and a right quasi-special order of curvature only to the abscissa of the right end of the characteristic polygonal line.*
5. *An order of curvature equal to the abscissa of the point of intersection of two special links is special.*
6. *If at the point of intersection of a special and an ordinary link more than two straight lines of the diagram intersect without coinciding with one another, then the corresponding order of curvature will be simultaneously ordinary and left quasi-special (if the point under consideration lies at the left end of the special link), or ordinary and right quasi-special (if this point lies at the right end of the special link).*
7. *If there exists an order of curvature corresponding to some point of a special link and not coinciding with any of the characteristic numbers, then it is special.*

We have found simple analytic criteria for distinguishing all the cases indicated. If the order of curvature is zero, infinite, special, or quasi-special, then there always exists an infinite set of characteristics entering the origin with this order. If the order of curvature is ordinary, then equation (3), after reduction, takes the form

$$x^{p/q}\frac{du}{dx} = \frac{P_{n_1}(u) + x^{s/q}\theta_1(x, x^{1/q}, u)}{Q_{p_1}(u) + x^{r/q}\theta_2(x, x^{1/q}, u)}, \tag{5}$$

where p, q, r, s are positive integers; θ_1 and θ_2 are analytic functions of $x, x^{1/q}$, and u; $P_{n_1}(u)$ and $Q_{p_1}(u)$ are polynomials whose degrees do not exceed, respectively, n and p. All measures of curvature for the given order of curvature are equal to the real, nonzero roots of the equation

$$P_{n_1}(u) = 0; \tag{6}$$

if equation (6) has no such roots, then equation (1) has no characteristics entering the origin with the given order of curvature. We shall distinguish two kinds of measures of curvature corresponding to an ordinary order of curvature: ordinary measures of curvature and special ones. For ordinary measures of curvature, $P_{n_1}(u) = 0$, but $Q_{p_1}(u) \neq 0$. For special measures of curvature, $P_{n_1}(u) = Q_{p_1}(u) = 0$. Let $u = \gamma$ be an ordinary measure of curvature. The substitutions $x^{1/q} = x_1$ and $u - \gamma = u_1$ bring equation (6) to the form

$$x_1^m \, du_1/dx_1 = f_0(x_1) + u_1 f_1(x_1) + u_1^2 f_2(x_1) + \cdots, \tag{7}$$

where the number m is an integer, positive, and the functions $f_1, f_2, \ldots$ are analytic in a neighborhood of the origin. If $f_0(0) \neq 0$, then equation (7) is a Briot-Bouquet equation, for which it is known ([3]) that either an infinite set of characteristics or only one characteristic enters the origin from the right, depending on whether $f_0(0) > 0$ or $f_0(0) < 0$. If $f_0(0) = 0$, then the lowest of the functions $f_i(x_1)$ containing a constant term has index n_1 ($n_1 \leq n$). In this case equation (7) is a generalized Briot-Bouquet equation, which, by means of a series of substitutions

$$u_1 = (u_2 - \gamma_2)x_1^{\lambda_2}, \quad u_2 = (u_3 - \gamma_3)x_1^{\lambda_3}, \ldots, u_{N-1} = (u_N - \gamma_N)x_1^{\lambda_N}, \tag{8}$$

(where the quantities $\lambda_2, \lambda_3, \ldots, \lambda_N$ and $\gamma_2, \gamma_3, \ldots, \gamma_N$ are successive orders and measures of curvature) is reduced either to such a generalized Briot-Bouquet equation for which there is a zero, infinite, special, or quasi-special order of curvature, or to such an equation for which none of the measures of curvature has real roots, or, finally, to a simple Briot-Bouquet equation. N. B. Khaimov proved ([4]) that the number N of transformations (8) is determined by the inequality

$$N \leq [m/2] \, f(n_1), \tag{9}$$

where the function of an integer argument $f(n)$ is defined as follows:

$$f(0) = 0, \qquad f(1) = 1, \qquad f(2) = 4,$$

$$f(n) = 4[n + n(n-2) + n(n-2)(n-4) + \cdots + n!!] + n!! \quad (n > 2). \tag{10}$$

Recently the author, jointly with D. M. Gruz, refined this result of N. B. Khaimov, showing that for $f(n)$ one may use the simpler formula $f(n) = 4n!!$.

Thus, for an ordinary order and an ordinary measure of curvature there exists either an infinite set of characteristics, or one characteristic, or no characteristic entering the origin from the right (or from the left) and having the given order and measure. The distinction between each of these three possibilities is established by a finite number of operations, an exact upper bound for which is given by formulas (9) and (10). If γ is a special measure of curvature, then the substitutions $x = x_1^q$, $u - \gamma = \bar{u}$ no longer bring equation (5) to the form (7), but lead to an equation of the same type as (5). The author, jointly with D. M. Gruz, proved that if the origin is an isolated singular point for equation (1), then after a finite number of substitutions of the form (8) (an exact upper bound for this number of substitutions has been found by us), either equation (1) is reduced to type (7), or it is found that no characteristics with definite tangents enter the origin. In particular, if the numerator and denominator of the right-hand side of equation (1) are polynomials of degree n, then the number of substitutions (8) does not exceed $n!/n^2$.

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Note: Figure translations are in progress. See original paper for figures.