REDUCTION OF AN EVOLUTIONARY SYSTEM OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS TO ONE EQUATION

A system of differential equations in partial derivatives with constant coefficients of the form

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. M. Borok

REDUCTION OF AN EVOLUTIONARY SYSTEM OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS TO ONE EQUATION

(Presented by Academician S. L. Sobolev on 17 XII 1956)

A system of differential equations in partial derivatives with constant coefficients of the form

$$\frac{\partial u(x,t)}{\partial t} = P\left(i\frac{\partial}{\partial x}\right) u(x,t); \tag{1}$$

is considered; here $u(x,t) = \{u_1(x,t), \dots, u_N(x,t)\}$, $x = \{x_1, \dots, x_n\}$; $P\left(i\frac{\partial}{\partial x}\right)$ is a square matrix of order N, whose elements are "polynomials" in the "variables" $i\frac{\partial}{\partial x_1}, \dots, i\frac{\partial}{\partial x_n}$ with constant coefficients.

Theorem 1. *A system of differential equations in partial derivatives of the form (1) is equivalent to one partial differential equation with constant coefficients of the form*

$$\frac{\partial^N u(x,t)}{\partial t^N} = \sum_{m=1}^{N} P_m\left(i\frac{\partial}{\partial x}\right) \frac{\partial^{N-m} u(x,t)}{\partial t^{N-m}} \tag{2}$$

(or to a system of several equations of the form (2), each of which is integrated independently of the others).

Proof. Let $E_1(\lambda), \dots, E_N(\lambda)$ be the invariant factors[1] of the matrix $P(s)$ ($s = \{s_1, \dots, s_n\}$), which is obtained from the matrix $P\left(i\frac{\partial}{\partial x}\right)$ by replacing the "variable" $i\frac{\partial}{\partial x_k}$ by s_k ($k = 1, \dots, n$), and let

$$E_j(\lambda) = \lambda^{k_j} - P_{1j}(s)\lambda^{k_j-1} - \dots - P_{k_j j}(s) \quad (j = 1, \dots, N;\ \sum_1^N k_j = N),$$

$P_{mj}(s)$ ($m = 1, \dots, k_j$) being a polynomial in $s_1, \dots, s_n$.

Suppose that $E_1(\lambda) = \dots = E_j(\lambda) = 1$, $E_{j+1}(\lambda) \neq 1$, and construct the quasi-diagonal matrix

$$Q(s) = \begin{Vmatrix} E_1 & & & & \\ & E_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & E_{N-j} \end{Vmatrix}; \tag{3}$$

here in the off-diagonal blocks all elements are equal to zero, while the diagonal block E_p is constructed according to the rule

$$E_p(s) = \begin{Vmatrix} 0 & 1 & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ P_{k_j+p,j+p}(s) & P_{k_j+p-1,j+p}(s) & \cdots & P_{2,j+p}(s) & P_{1,j+p}(s) & \end{Vmatrix}. \tag{4}$$

As is known ([1]), the matrices $P(s)$ and $Q(s)$ have the same invariant factors and therefore are similar. Consequently, there exists a matrix $T_1(s)$ ($\det T_1(s) \neq 0$) such that $Q(s) = T_1(s)P(s)T_1^{-1}(s)$. Moreover, the matrix $T_1(s)$ can be chosen so that its elements are polynomials in $s_1, \dots, s_n$.

Now put $v(x,t) = T_1\left(i\frac{\partial}{\partial x}\right) u(x,t)$, where the matrix $T_1\left(i\frac{\partial}{\partial x}\right)$ is obtained from the matrix $T_1(s)$ by replacing each s_j by $i\frac{\partial}{\partial x_j}$ $(j = 1, \dots, n)$. If the vector-function $u(x,t)$ was a solution of system (1), then the vector-function $v(x,t)$ will be a solution of the system

$$\frac{\partial v(x,t)}{\partial t} = Q\left(i\frac{\partial}{\partial x}\right) v(x,t). \tag{5}$$

Thus, applying the differential operator $T_1\left(i\frac{\partial}{\partial x}\right)$ to a solution $u(x,t)$ of system (1), we obtain a solution $v(x,t)$ of system (5).

Let $v_0(x,t)$ be some solution of system (5). Find a solution $u(x,t)$ of the system $T_1\left(i\frac{\partial}{\partial x}\right) u = v_0(x,t)$; such a solution always exists (in generalized functions) ([2]). Then $u(x,t)$ will be a solution of system (1). Thus, each solution of system (5) is obtained as a result of applying the differential operator $T_1\left(i\frac{\partial}{\partial x}\right)$ to some solution of system (1).

In an analogous way, representing the matrix $Q(s)$ in the form $Q = T_2^{-1}PT_2$, where the elements of the matrix $T_2(s)$ are polynomials in $s_1, \dots, s_n$, we are convinced of the validity of the converse assertion: applying to a solution $v(x,t)$ of system (5) the operator $T_2\left(i\frac{\partial}{\partial x}\right)$ (the matrix $T_2\left(i\frac{\partial}{\partial x}\right)$ is obtained from the matrix $T_2(s)$ by replacing each s_j by $i\frac{\partial}{\partial x_j}$ $(j = 1, \dots, n)$), we obtain a solution $u(x,t)$ of system (1), and each solution $u(x,t)$ of system (1) is obtained by applying the operator $T_2\left(i\frac{\partial}{\partial x}\right)$ to some solution $v(x,t)$ of system (5). Thus, system (1) is equivalent to system (5) (in the class of generalized functions). In view of the form of the matrix $Q(s)$, from (3) and (4) we obtain that system (5) decomposes into $N-j$ systems, each of which is integrated independently of the others and has the form

$$\frac{\partial v_{(p)}(x,t)}{\partial t} = E_p\left(i\frac{\partial}{\partial x}\right) v_{(p)}(x,t), \tag{6}$$

where

$$v_{(p)}(x,t) = \{v_{k_j+1+\dots+k_{j+p-1}+1}(x,t), \dots, v_{k_j+1+\dots+k_{j+p}}(x,t)\}.$$

System (6) with respect to the vector-function $v_{(p)}(x,t)$ is, obviously, equivalent to one equation with respect to the function $w_p(x,t) = v_{k_j+1+\dots+k_{j+p-1}+1}(x,t)$:

$$\frac{\partial^{k_j+p} w_p(x,t)}{\partial t^{k_j+p}} = \sum_{m=1}^{k_j+p} P_{m,j+p}\left(i\frac{\partial}{\partial x}\right)\frac{\partial^{k_j+p-m} w_p(x,t)}{\partial t^{k_j+p-m}},$$

which completes the proof of the theorem.

The system (1) is called **hyperbolic** (3) if the characteristic roots $\lambda_1(s_1,\dots,s_n),\dots,\lambda_N(s_1,\dots,s_n)$ $(s_k = \sigma_k + i\tau_k,\ k = 1,\dots,n)$ of the matrix $P(s)$ satisfy the conditions

$$\max_{1\le j\le N} \operatorname{Re}\lambda_j(s_1,\dots,s_n) \le A_1|s| \quad \text{for} \quad |s| = \left(\sum_{j=1}^{n}|s_j|^2\right)^{1/2} \ge 1,$$

$$\max_{1\le j\le N} \operatorname{Re}\lambda_j(\sigma_1,\dots,\sigma_n) \le A_2. \tag{7}$$

Theorem 2. *If the system (1) is hyperbolic, then the Cauchy problem for it can be reduced to the Cauchy problem for a first-order system.*

The proof of Theorem 2 is based on the following lemma:

Lemma. *Let*

$$R(\lambda) = \lambda^N - R_1(s)\lambda^{N-1} - \dots - R_N(s)$$

be a polynomial in λ, whose coefficients $R_j(s) = R_j(s_1,\dots,s_n)$ $(j = 1,\dots,N)$ are polynomials in the complex variables $s_k = \sigma_k + i\tau_k$ $(k = 1,\dots,n)$; let $\lambda_j(s) = \lambda_j(s_1,\dots,s_n)$ $(j = 1,\dots,N)$ be its roots.

Then, if for $|s| = (|s_1|^2 + \dots + |s_n|^2)^{1/2} \ge 1$

$$\max_{1\le j\le N} \operatorname{Re}\lambda_j(s) \le C_1|s|^k, \tag{8}$$

then for $|s| \ge 1$

$$\max_{1\le j\le N} |\lambda_j(s)| \le C_2|s|^k. \tag{8'}$$

Proof of Theorem 2. The Cauchy problem for the system (1) with initial condition $u(x,0)=u_0(x)$, as follows from Theorem 1, can be reduced to the Cauchy problem for a system of equations of the form

$$\frac{\partial^{n_k} v_k(x,t)}{\partial t^{n_k}} = \sum_{m=1}^{n_k} P_{mk}\left(i\frac{\partial}{\partial x}\right)\frac{\partial^{n_k-m} v_k(x,t)}{\partial t^{n_k-m}}$$

$$\left(k=1,\dots,l; \qquad \sum_{k=1}^{l} n_k = N\right) \tag{9}$$

with initial condition

$$\left.\frac{\partial^p v_k(x,t)}{\partial t^p}\right|_{t=0} = v_{kp}(x) \quad (p=0,\dots,n_k-1;\ k=1,\dots,l),$$

where the function $v_0(x)$ must be a solution of the system

$$T_2\left(i\frac{\partial}{\partial x}\right)v_0 = u_0,$$

where

$$v_0(x)=\{v_{1,0}(x),\dots,v_{1,n_1-1}(x),\dots,v_{l,n_l-1}(x)\}; \quad u_0(x)=\{u_{1,0}(x),\dots,u_{N,0}(x)\}.$$

We shall show that each of the equations (9) is an equation of Kowalevsky type, i.e., the degree of the polynomial $P_{mk}(s)$ does not exceed the number m.

Let, as in the proof of Theorem 1,

$$E_j(\lambda)=\lambda^{k_j}-P_{1j}(s)\lambda^{k_j-1}\cdots-P_{k_j j}(s)=(\lambda-\lambda_1(s))^{l_{j1}}\cdots(\lambda-\lambda_{t_j}(s))^{l_{jt_j}};$$

here $j=1,\dots,N$;

$$1\le t_j\le N; \qquad k_j=\sum_{i=1}^{t_j} l_{ji}; \qquad \sum_{j=1}^{N} k_j = N;$$

$\lambda_r(s)$ $(r=1,\dots,N)$ are the characteristic roots of the matrix $P(s)$ (among them there may be equal ones). Then

$$P_{mj}(s)=(-1)^{m-1}\sum_{\substack{1\le j_r\le t_j\ (1\le r\le m),\\ j_i\ne j_l\ (i\ne l)}} \lambda_{j_1}(s)\cdots\lambda_{j_m}(s). \tag{10}$$

By virtue of the hyperbolicity of the system (1) and the lemma, for $|s| \geq 1$ the estimate holds

$$\max_{1\leq j\leq N} |\lambda_j(s)| \leq A_3|s|. \tag{11}$$

Then from (10) and (11) it follows that, for $|s| \geqslant 1$, $|P_{mj}(s)| \leqslant A_4|s|^m$. Hence the degree of the polynomial $P_{mj}(s)$ in the aggregate of variables $s_1, \dots, s_n$ is not greater than m, and, consequently, the order of the differential operator

$$P_{mj}\left(i\frac{\partial}{\partial x}\right) = P_{mj}\left(i\frac{\partial}{\partial x_1}, \dots, i\frac{\partial}{\partial x_n}\right)$$

also does not exceed m. Thus each equation of the system (9) is an equation of Kovalevskaya type, and for the latter the assertion of the theorem is valid (4).

Proof of the lemma. Since $\lambda_j(s)$ $(j = 1, \dots, N)$ are the roots of the polynomial $R(\lambda)$, we have

$$R_1(s) = \sum_{j=1}^{N} \lambda_j(s), \dots, \quad R_N(s) = \prod_{j=1}^{N} \lambda_j(s).$$

Introduce the notation $(j = 1, \dots, N)$:

$$\operatorname{Re} R_j(s) = L_j(s) = L_j(\sigma_1, \dots, \sigma_n, \tau_1, \dots, \tau_n);$$

$$\operatorname{Im} R_j(s) = T_j(s) = T_j(\sigma_1, \dots, \sigma_n, \tau_1, \dots, \tau_n);$$

$$\operatorname{Re} \lambda_j(s) = \mu_j(s) = \mu_j(s_1, \dots, s_n), \quad \operatorname{Im} \lambda_j(s) = \nu_j(s) = \nu_j(s_1, \dots, s_n).$$

Let

$$M(S) = \max_{1\leqslant j\leqslant N} \nu_j(s) \quad \text{for} \quad \max_{1\leqslant j\leqslant n} |s_j| \leqslant S.$$

The formula (5)*

$$M(S) = aS^{k_1}(1 + o(1)), \tag{12}$$

holds, where either $a = 0$, or $a \neq 0$ is a real number, k_1 is a rational number, and $o(1) \to 0$ as $S \to \infty$.

Suppose $k_1 > k$ and choose k_2 so that

$$k_1 > k_2 > \frac{k + (N-1)k_1}{N} > k.$$

In view of (12), among the functions $\nu_j(s)$ $(j = 1, \dots, N)$ there will be several functions $\nu_{j_1}(s), \dots, \nu_{j_q}(s)$ which, on some sequence of points

$$s_p = (s_{1p}, \dots, s_{np}) \quad (|s_p| \xrightarrow[p\to\infty]{} \infty)$$

satisfy the estimate

$$|\nu_{j_r}(s_p)| > p|s_p|^{k_2} \quad (r = 1, \dots, q). \tag{13}$$

Suppose that $q < N$ and is even. Then from (8), (12), and (13) one can conclude that the polynomials $L_q(s)$ and $T_q(s)$ have different degrees: the degree of $L_q(s)$ is not less than qk_2, while the degree of $T_q(s)$ is not greater than

$$k + (q-1)k_1 < qk_2.$$

The polynomials $L_q(s)$ and $T_q(s)$, as the real and imaginary parts of one and the same polynomial $R_q(s)$, must have the same degree. Therefore there exists a function $\nu_{j_{q+1}}(s)$ satisfying the estimate (13). Thus all functions $\nu_j(s)$ ($j = 1, \dots, N$) satisfy (13). Then $L_N(s)$ has degree not less than Nk_2, while the degree of $T_N(s)$ does not exceed

$$k + (N-1)k_1 < Nk_2.$$

Since the polynomials

$$L_N(s) = L_N(\sigma_1, \dots, \tau_n) \quad \text{and} \quad T_N(s) = T_N(\sigma_1, \dots, \tau_n)$$

must have the same degree, a contradiction is obtained, proving that $k_1 \leqslant k$. (The reasoning has been given for even N; for odd N the polynomials $L_N(s)$ and $T_N(s)$ should be interchanged.) Now (8) and (12) imply (8′), as was required.

The author takes this opportunity to express his deep gratitude to Prof. G. E. Shilov for his scientific guidance.

Moscow State University
named after M. V. Lomonosov

Received
14 XII 1956

References

1. A. I. Maltsev, *Foundations of Linear Algebra*, 1948.
2. L. Ehrenpreis, *Am. J. Math.*, **76**, No. 4, 883 (1954).
3. I. M. Gelfand, G. E. Shilov, *Uspekhi Mat. Nauk*, **8**, issue 6 (1953).
4. I. G. Petrovskii, *Lectures on Partial Differential Equations*, 1953.
5. L. Gårding, *Acta Math.*, **85**, Nos. 1-2 (1951).

* In (5), formula (12) is proved for

$$M_1(S) = \max_{1 \leqslant j \leqslant N} \mu_j(s) \quad \text{for} \quad \max_{1 \leqslant j \leqslant N} |s_j| \leqslant S;$$

in our case the proof is analogous.

Note: Figure translations are in progress. See original paper for figures.