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Abstract

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SIMILARITY INTEGRALS OF HYDRODYNAMICS FOR HETEROGENEOUS AND HOMOGENEOUS PROCESSES

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Schemes of motion of a viscous gas accompanied by heterogeneous or homogeneous chemical reactions are encountered in many areas of practice. In the present work a number of new similarity integrals are obtained for systems of differential equations for the above-mentioned problems.

Similarity integrals of vortical and concentration fields for heterogeneous reactions

Let $u_x(x, y)$ and $u_y(x, y)$ be the velocity components of a plane viscous gas flow past a curvilinear profile; $c(x, y)$, the concentration; $p(x, y)$, the pressure; and μ, the dynamic coefficient of viscosity. Since in the problems considered here the velocities of motion are very far from sonic velocities, compressibility is not taken into account and one may set $\rho = 1$. Then the system of hydrodynamic-diffusion equations has the form:

$$u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right), \tag{1}$$

$$u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y} = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right), \tag{2}$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \tag{3}$$

$$u_x \frac{\partial c}{\partial x} + u_y \frac{\partial c}{\partial y} = \mu \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right). \tag{4}$$

(for gases the diffusion number $\mathrm{Pr} = \dfrac{D}{\mu}$ is usually taken to be equal to unity).

For the system (1)–(4) only one integral is known, $c = a u_x + b$, which holds only for the special case of absence of a pressure gradient (the boundary layer on a plate). Here we shall obtain an integral for the general case with the presence of a pressure gradient (flow past a curvilinear profile).

Introduce a new dependent variable

$$\Omega = c - a\left(\frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x}\right), \tag{5}$$

where a is a certain constant. By virtue of (1), (2), and (4) we obtain the following equation for Ω:

$$u_x\frac{\partial \Omega}{\partial x} + u_y\frac{\partial \Omega}{\partial y} = \mu\left(\frac{\partial^2 \Omega}{\partial x^2} + \frac{\partial^2 \Omega}{\partial y^2}\right). \tag{6}$$

This equation has the integral

$$\Omega = b = \text{const}, \tag{7}$$

or, otherwise,

$$c = a\omega + b, \tag{7'}$$

i.e., we obtain the similarity integral of vortex and concentration fields (for flow around a profile of arbitrary shape).

Let us determine what boundary conditions correspond to integral $(7')$. For practical applications it is most convenient to use the boundary-layer scheme. At the boundary of the layer $(y = \infty)$ we have $c = \bar{c}$; $\omega = 0$; consequently, $b = \bar{c}$. Further, according to (1) (written in boundary-layer form) for the vorticity gradient, we have the boundary condition on the surface of the profile:

$$\left(\frac{\partial \omega}{\partial y}\right)_0 = \frac{1}{\mu}\frac{dp}{dx}. \tag{8}$$

For the concentration gradient in the case under consideration of a surface reaction we have

$$\left(\frac{\partial c}{\partial y}\right)_0 = q(x). \tag{8'}$$

Here $q(x)$ is a prescribed function determining the law of solubility of the surface.

Comparison of $(7')$, (8), and $(8')$ gives the relation between $q(x)$ and $\dfrac{dp}{dx}$:

$$q(x) = \frac{a}{\mu}\frac{dp}{dx}. \tag{9}$$

For example, for gas flow in a plane channel $\left(\frac{dp}{dx} = \text{const}\right)$ we obtain $q(x) =$ const, and since $u_x = \bar{u}_x\left(1 - \frac{y^2}{h^2}\right)$ (where $2h$ is the channel width, $\bar{u}_x$ is the axial velocity), integral (7′) gives $c = \bar{c} - \frac{2a\bar{u}_x}{h^2}y$.

Analogous solutions are obtained for gas flow accompanied by a surface reaction, and in other cases, for example for a confuser, polygonal and power-law distributions of velocities, etc.

Generalized similarity integral

We shall show that, in the particular case of flow past a plate, a more general integral of equation (6) can also be obtained. In this case, in equation (1) (written in boundary-layer form), one must set $\frac{dp}{dx} = 0$, and the integral has the form (k and b are constants):

$$\Omega = ku_x + b, \tag{10}$$

or, otherwise,

$$c = a\omega + ku_x + b. \tag{11}$$

At the boundary of the layer $c = \bar{c}$; $u_x = \bar{u}_x$, $\omega = 0$, and, consequently, $b = \bar{c} - k\bar{u}_x$. Further, on the surface we have $\left(\frac{\partial\omega}{\partial y}\right)_0 = 0$; $\left(\frac{\partial c}{\partial y}\right)_0 = q(x)$. Consequently, according to (11), $q(x) = k\left(\frac{\partial u_x}{\partial y}\right)_0$, and since u_x is determined by the Blasius formula in the form

$$u_x = \bar{u}_x f\left(y\sqrt{\frac{\bar{u}_x}{\mu x}}\right),$$

where f is a known function, then

$$q(x) = \frac{m}{\sqrt{x}}, \tag{12}$$

where

$$m = kf'(0)\bar{u}_x\sqrt{\frac{\bar{u}_x}{\mu}} = 0.332\,k\bar{u}_x\sqrt{\frac{\bar{u}_x}{\mu}}.$$

Similarity integral for homogeneous reactions

In the case of a homogeneous (volume) reaction, the equation for concentrations (written in boundary-layer form) has the form

$$u_x\frac{\partial c}{\partial x} + u_y\frac{\partial c}{\partial y} = \mu\frac{\partial^2 c}{\partial y^2} + V_{ch}. \tag{13}$$

Here V_{ch} is the rate of the volumetric chemical reaction, which in the general case may be represented as a second-degree polynomial with respect to the concentration:

$$V_{ch} = a_0 + a_1c + a_2c^2. \tag{14}$$

The most important special cases here are a first-order reaction ($a_0 = a_2 = 0$) and a second-order reaction ($a_0 = a_1 = 0$).

The equation of motion (1) retains its form:

$$u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial u} = \mu\frac{\partial^2 u_x}{\partial y^2} + \overline{u_xu_x'} \tag{15}$$

(here, according to the integral at the boundary of the boundary layer, $-\dfrac{\partial p}{\partial x}$ has been replaced by $\overline{u_xu_x'}$; the overbar corresponds to the boundary of the layer).

We shall show that the solution of the system (13)–(15) can be found in the form of a similarity integral of the concentration and velocity fields:

$$c = c_0(x) + \frac{\overline{c} - c_0(x)}{\overline{u}_x}\,u_x \tag{16}$$

(the index 0 corresponds to the surface of the streamlined profile).

Substituting (16) into (13), using (15), and also introducing the variable $r = \overline{c} - c_0(x)$, we obtain, by comparing the coefficients of like powers of u_x, a system of equations for determining $\overline{u}_x(x)$ and r:

$$\overline{u}_x'r = a_2r^2 - (a_1 + 2a_2\overline{c})r + V_{ch}, \tag{17}$$

$$-\overline{u}_xr' = 2a_2(\overline{c} - r)r + a_1r, \tag{18}$$

$$\overline{u}_x r' = \overline{u}_x' r + a_2 r^2. \tag{19}$$

Adding (17), (18), and (19), we obtain the obvious condition

$$\overline{V}_{ch} = 0, \tag{20}$$

corresponding to the absence of a homogeneous reaction at the boundary of the layer. Introducing also the notation

$$a_1 + 2a_2\overline{c} = k = \left(\frac{dV_{ch}}{dc}\right)_{c=\overline{c}},$$

we have (we write out only the first two equations, since the third is their consequence):

$$\overline{u}_x' = a_2 r - k, \tag{21}$$

$$-\overline{u}_x r' = (k - 2a_2 r)r. \tag{22}$$

The simultaneous solution of the system (21), (22) gives:

a) for $a_2 = 0$ $(k = a_1)$

$$\overline{u}_x = -a_1 x + c_2; \qquad r = c_1(-a_1 x + c_2), \tag{23}$$

which corresponds to a polygonal velocity distribution;

b) for $a_2 \neq 0$

$$\overline{u}_x = -\frac{k}{2}(x + c_1) + \frac{c_2}{x + c_1}, \tag{24}$$

$$r = \overline{c} + \frac{a_1}{2a_2} - \frac{c_2}{a_2(x + c_1)^2}, \tag{24'}$$

which corresponds to a combination of polygonal and hyperbolic velocity distributions.

Homogeneous second-order reaction during flow in a confusor

In the case of a second-order reaction $a_0 = a_1 = 0$; $V_{ch} = a_2 c^2$, and, since $\overline{V}_{ch} = 0$, we also have $\overline{c} = 0$. Consequently, $k = a_1 + 2a_2\overline{c} = 0$, and formula (24) gives

$$\overline{u}_x = -\frac{c_2}{x + c_1}, \tag{25}$$

i.e., flow in a confusor.

In this case (24′) determines the value of the concentration at the surface in the form

$$c_0 = \overline{c} - r = \frac{c_2}{a_2(x + c_1)^2}.$$

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