HYDROMECHANICS

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1957

SovietRxiv

Abstract

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AN ASYMPTOTIC SOLUTION OF THE EQUATIONS OF ONE-DIMENSIONAL UNSTEADY MOTION OF AN IDEAL GAS WITH CYLINDRICAL SYMMETRY

(Presented by Academician L. I. Sedov, 16 IV 1957)

The first asymptotic solution of the problem of the attenuation of shock waves was found by L. D. Landau ([1]); he obtained the first term of the solution for the spherical and cylindrical cases. L. I. Sedov ([2]) solved this problem in characteristic variables; he also obtained the first term. Yu. L. Yakimov ([3]) developed L. I. Sedov' s method and, in the spherical case, obtained an asymptotic solution with terms of order of smallness higher than the first.

In the present work, by an analogous method, the solution is found in the cylindrical case; then, with the aid of this solution, the asymptotic laws of attenuation of shock waves are investigated; terms of order of smallness higher than the first are found.

1. A one-dimensional unsteady flow of an ideal perfect gas with cylindrical symmetry must be described by the system of equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \qquad \frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial r} + \frac{\rho u}{r} = 0, \qquad \frac{\partial}{\partial t}\frac{p}{\rho^\gamma} + u\frac{\partial}{\partial r}\frac{p}{\rho^\gamma} = 0, \quad (1)$$

where r is the coordinate (radius); t is time; u is velocity; ρ is density; p is pressure.

It is known that this system has three families of characteristic directions

$$dr = (a+u)\,dt, \qquad dr = (-a+u)\,dt, \qquad dr = u\,dt$$

(a is the speed of sound).

We shall carry out computations and arguments analogous to those carried out in ([3]). In the coordinates r, ξ_1

$$\mu_{10}(r,\xi_1)\,d\xi_1 = dr - (a+u)\,dt \tag{2}$$

we shall find the wave going away from the axis of cylindrical symmetry, or the direct wave, if p, ρ, u are sought in the form of series in powers of $\ln^i r / r^{k/2}$ (k varies from 1 to ∞; i, from 0 to $k-1$).

In the coordinates r, ξ_2

$$\mu_{20}(r,\xi_2)\,d\xi_2 = dr - (-a+u)\,dt \tag{3}$$

we shall find the wave going toward the axis of cylindrical symmetry, or the reflected wave. Then we combine these two waves by an interaction wave, which disappears when one of the first two waves disappears.

Thus we obtain an asymptotic solution containing explicitly two arbitrary functions and a countable set of arbitrary constants. Let us write out the first terms of this solution:

$$p = 1 + [F_{10}(\xi_1) + f_{10}(\xi_2)]\,\frac{1}{\sqrt{r}} + [F_{20}(\xi_1) + f_{20}(\xi_2) + \alpha_{20}(\xi_1,\xi_2)]\,\frac{1}{r} + \cdots,$$

$$\begin{aligned} \rho = 1 + [\Psi_{10}(\xi_1) + \psi_{10}(\xi_2)]\,\frac{1}{\sqrt{r}} + [\Psi_{21}(\xi_1) + \psi_{21}(\xi_2)]\,\frac{\ln r}{r} + \\ + [\Psi_{20}(\xi_1) + \psi_{20}(\xi_2) + \beta_{20}(\xi_1,\xi_2)]\,\frac{1}{r} + \cdots, \end{aligned} \tag{4}$$

$$\begin{aligned} u = [\Phi_{10}(\xi_1) + \varphi_{10}(\xi_2)]\,\frac{1}{\sqrt{r}} + [\Phi_{21}(\xi_1) + \varphi_{21}(\xi_2)]\,\frac{\ln r}{r} + \\ + [\Phi_{20}(\xi_1) + \varphi_{20}(\xi_2) + \omega_{20}(\xi_1,\xi_2)]\,\frac{1}{r} + \cdots, \end{aligned}$$

where

$$\Phi_{10} = \frac{1}{\sqrt{\gamma}}F_{10}, \qquad \Psi_{10} = \frac{1}{\gamma}F_{10} + C_{\Psi_{10}}, \qquad \mu_{10} = \left(1 + \frac{1}{\gamma}\right)F'_{10}(\xi_1)\sqrt{r} + 1 + \cdots,$$

$$F_{21} = 0, \qquad \Phi_{21} = C_{\Phi_{21}}, \qquad \Psi_{21} = C_{\Psi_{21}}, \qquad F_{20} = \frac{\gamma-3}{4\gamma}F_{10}^2 + \frac{1}{4}C_{\Psi_{10}}F_{10} + \sqrt{\gamma}C_{\Phi_{21}}, \tag{5}$$

$$\Phi_{20} = -\frac{\gamma+5}{4\gamma\sqrt{\gamma}}F_{10}^2 - \frac{1}{4\sqrt{\gamma}}C_{\Psi_{10}}F_{10} + C_{\Phi_{20}},$$

$$\Psi_{20} = -\frac{1+\gamma}{8\gamma^2}F_{10}^2 + \frac{5}{4\gamma}C_{\Psi_{10}}F_{10} + C_{\Psi_{20}}.$$

direct wave;

$$\varphi_{10} = -\frac{1}{\sqrt{\gamma}}f_{10}, \qquad \psi_{10} = \frac{1}{\gamma}f_{10} + c_{\psi_{10}}, \qquad f_{21} = 0, \qquad \varphi_{21} = c_{\varphi_{21}},$$

$$\psi_{21} = c_{\psi_{21}}, \qquad \mu_{20}(r,\xi_2) = \left(1+\frac{1}{\gamma}\right)f'_{10}(\xi_2)\sqrt{r} + 1 + \cdots, \tag{6}$$

$$f_{20} = \frac{\gamma-3}{4\gamma}f_{10}^2 + \frac{1}{4}c_{\psi_{10}}f_{10} + c_{f_{20}}, \qquad \varphi_{20} = \frac{\gamma+5}{4\gamma\sqrt{\gamma}}f_{10}^2 + \frac{1}{4\sqrt{\gamma}}C_{\Psi_{10}}f_{10} + c_{\varphi_{20}},$$

$$\psi_{20} = -\frac{1+\gamma}{8\gamma^2}f_{10}^2 + \frac{5}{4\gamma}C_{\Psi_{10}}f_{10} + c_{\psi_{20}}, \qquad c_{f_{20}} = -\sqrt{\gamma}c_{\varphi_{21}}$$

reverse wave;

$$\mu_1 = \left(1+\frac{1}{\gamma}\right)F'_{10}(\xi_1)\sqrt{r} + 1 + \cdots, \qquad \mu_2 = \left(1+\frac{1}{\gamma}\right)f'_{10}(\xi_2)\sqrt{r} + 1 + \cdots,$$

$$\alpha_{10} = \beta_{10} = \omega_{10} = 0, \qquad \alpha_{21} = \beta_{21} = \omega_{21} = 0,$$

$$\beta_{20} = -\frac{\gamma-1}{\gamma^2}F_{10}f_{10} + \frac{5}{4\gamma}\left(c_{\psi_{10}}F_{10} + C_{\Psi_{10}}f_{10}\right),$$

$$\alpha_{20} = \frac{1}{4}\left(c_{\psi_{10}}F_{10} + C_{\Psi_{10}}f_{10}\right), \qquad \omega_{20} = -\frac{\gamma+1}{\gamma\sqrt{\gamma}}F_{10}f_{10} - \frac{1}{4\sqrt{\gamma}}\left(c_{\psi_{10}}F_{10} + C_{\psi_{10}}f_{10}\right) \tag{7}$$

interaction wave.

2. Let us apply the solution obtained to the investigation of the question of the attenuation of shock waves. Suppose that the discontinuity moves in an undisturbed medium $p = 1,\ \rho = 1,\ u = 0$. Then the conditions at the discontinuity can be written in the form

$$\Delta p \Delta \rho + \frac{2}{\gamma - 1}(\gamma \Delta \rho - \Delta p) = 0, \qquad u^2 = \frac{1}{\rho} \Delta p \Delta \rho, \qquad D^2 = \rho \frac{\Delta p}{\Delta \rho}. \tag{8}$$

Here $\Delta p = p - 1$; $\Delta \rho = \rho - 1$; $D(r)$ is an as yet unknown function; u is the velocity of motion of the front of the shock wave.

Substituting the first three terms of the solution (4) into (8), we obtain, respectively,

$$C_{\Psi_{10}} = c_{\psi_{10}} = C_{\Psi_{21}} = c_{\psi_{21}} = C_{\Phi_{21}} = c_{\varphi_{21}} = C_{\Psi_{20}} = c_{\psi_{20}} = C_{\Phi_{20}} = c_{\varphi_{20}} = 0; \tag{9}$$

$$f_{10}(\xi_2) = -\frac{\gamma - 3}{8\gamma} F_{10}^2(\xi_1) \frac{1}{\sqrt{r}} + o(r^{\varepsilon - 1}); \tag{10}$$

$$D = \sqrt{\gamma} \left\{ 1 + \frac{\gamma + 1}{4\gamma} \left[F_{10}(\xi_1) + f_{10}(\xi_2) \right] \frac{1}{\sqrt{r}} \right\} + o(r^{\varepsilon - 1}). \tag{11}$$

Thus, assuming that $F_{10}(\xi_1)$ is the prescribed shape of the incident wave, we determine the remaining functions.

We investigate the asymptotic behavior of the shock wave by means of the equality

$$\mu_1 d\xi_1 = dr - (a + u)\, dt. \tag{12}$$

At the discontinuity $dr/dt = D(r)$; therefore (12) at the discontinuity takes the form

$$\mu_1 d\xi_1 = \left[1 - \frac{a + u}{D} \right] dr. \tag{13}$$

Substituting the expressions for μ_1, $(a + u)/D$ in terms of series, and taking (9) –(11) into account, we obtain a differential equation valid at the discontinuity

$$\frac{dF_{10}}{dr} \left[\left(1 + \frac{1}{\gamma} \right) \sqrt{r} + \frac{d\xi_1}{dF_{10}} + \cdots \right] = -\frac{1}{4} \left(1 + \frac{1}{\gamma} \right) F_{10} \frac{1}{\sqrt{r}} + \cdots. \tag{14}$$

From this one can determine $F_{10}(r)$ at the discontinuity, if the derivative $d\xi_1/dF_{10}$ is found. The wave shape $F_{10}(r, t_0)$ at the initial instant is given to us (Fig. 1, $t = t_0$). Suppose that $F_{10}(r_0, t_0) = 0$, $\partial F_{10}/\partial r\big|_{r_0, t_0} \neq 0$, and that $F_{10}(r, t_0)$ is expandable in a Taylor series in powers of $r - r_0$.

Taking into account that $F_{10} = F_{10}(\xi_1)$ and, consequently, $\xi_1 = \xi_1(F_{10})$, and knowing $F_{10}(r, t_0)$, we find $\xi_1(F_{10})$ at the initial instant of time $t = t_0$, $dt = 0$.

From (13), for $t = t_0$, it follows that

$$\frac{\partial \xi_1}{\partial r}\bigg|_{r,t_0} = 1 - \left(1 + \frac{1}{\gamma}\right) \frac{\partial F_{10}}{\partial r}\bigg|_{r,t_0} \sqrt{r} + \cdots \tag{15}$$

Substituting into the right-hand side of (15) the expansions of $\partial F_{10}/\partial r\big|_{r,t_0}$, $\sqrt{r}$, ... in powers of $r - r_0$, and then replacing $r - r_0$ by the inverse expansion in powers of F_{10}, we obtain

$$\xi_1(F_{10}) = \xi_0 + \varkappa_1 F_{10} + \tfrac{1}{2}\varkappa_2 F_{10}^2 + \cdots, \tag{16}$$

where

$$\varkappa_1 = \frac{1}{\partial F_{10}/\partial r\big|_{r_0,t_0}} - \left(1 + \frac{1}{\gamma}\right)\sqrt{r_0},$$

$$\varkappa_2 = -\frac{\partial^2 F_{10}/\partial r^2\big|_{r_0,t_0}}{\left(\partial F_{10}/\partial r\big|_{r_0,t_0}\right)^3} - \frac{(1 + 1/\gamma)}{2\sqrt{r_0}} \frac{1}{\partial F_{10}/\partial r\big|_{r_0,t_0}}, \ldots$$

The dependence $\xi_1(F_{10})$ does not change with time. Hence, at any instant of time and for any radius,

$$\frac{d\xi_1}{dF_{10}} = \varkappa_1 + \varkappa_2 F_{10} + \cdots \tag{17}$$

Then (14) takes the form

$$\frac{dF_{10}}{dt}\left[\left(1 + \frac{1}{\gamma}\right)\sqrt{r} + \varkappa_1 + \varkappa_2 F_{10} + \cdots\right] = -\frac{1}{4}\left(1 + \frac{1}{\gamma}\right) F_{10} \frac{1}{\sqrt{r}} + \cdots. \tag{18}$$

The solution of (18) can be sought in the form

$$F_{10}(r) = \sum_{k=1}^{\infty} \sum_{i=-\infty}^{\infty} r^{-k/4} \ln^i r \cdot a_{ki} \qquad (i < \tfrac{1}{2}k).$$

From (18) we obtain $F_{10}(r)$ at the discontinuity as a function of the radius

Fig. 1

Figure 1: Fig. 1

$$F_{10}(r) = \frac{c_0}{r^{1/4}} - \frac{c_0}{2}\frac{\chi_2}{1+1/\gamma}\frac{1}{r^{3/4}} + O\left(r^{\varepsilon-1}\right) \tag{19}$$

(ε is an arbitrarily small positive quantity, c_0 an arbitrary constant).

Let us now find the form of the wave at any fixed instant of time t. We assume that there exists a point r_2, where $F_{10}(r_2, t) = 0$ (Fig. 1, t), and $F_{10}(r, t)$ is expanded in the series

Fig. 1

$$F_{10}(r,t) = \left.\frac{\partial F_{10}}{\partial r}\right|_{r_2,t}(r-r_2) + \frac{1}{2}\left.\frac{\partial^2 F_{10}}{\partial r^2}\right|_{r_2,t}(r-r_2)^2 + \cdots. \tag{20}$$

Substituting (20) into (18), taking into account $dt = 0$, we determine $\partial F_{10}/\partial r|_{r_2,t}$, $\partial^2 F_{10}/\partial r^2|_{r_2,t}$, ... as functions of the initial data and the radius

$$\left.\frac{\partial F_{10}}{\partial r}\right|_{r_2,t} = \frac{1}{(1+1/\gamma)\sqrt{r_2}+\chi_1}, \qquad \left.\frac{\partial^2 F_{10}}{\partial r^2}\right|_{r_2,t} = -\frac{1}{2}\frac{(1+2\chi_2+1/\gamma)\sqrt{r_2}+\chi_1}{\left[(1+1/\gamma)\sqrt{r_2}+\chi_1\right]^3}, \ldots \tag{21}$$

The asymptotic laws of decay of shock waves are obtained by substituting the results (9), (10), (19), and (20) into formula (4), and also into these same formulas after prior differentiation with respect to r:

$$p(r,t) = p(r_2,t) + \left.\frac{\partial p}{\partial r}\right|_{r_2,t}(r-r_2) + \frac{1}{2}\left.\frac{\partial^2 p}{\partial r^2}\right|_{r_2,t}(r-r_2)^2 + \cdots,$$

similarly for $\rho(r,t)$ and $u(r,t)$. At the point r_2, t, $p = \rho = 1$, $u = 0$,

$$\left.\frac{\partial p}{\partial r}\right|_{r_2,t} = \frac{1}{\sqrt{r_2}\left[(1+1/\gamma)\sqrt{r_2}+\chi_2\right]} + O\left(r_2^{\varepsilon-2}\right),$$

$$\left.\frac{\partial \rho}{\partial r}\right|_{r_2,t} = \frac{1}{\gamma}\left.\frac{\partial p}{\partial r}\right|_{r_2,t}, \qquad \left.\frac{\partial u}{\partial r}\right|_{r_2,t} = \frac{1}{\sqrt{\gamma}}\left.\frac{\partial p}{\partial r}\right|_{r_2,t},$$

$$\left.\frac{\partial^2 p}{\partial r^2}\right|_{r_2,t} = \frac{(2+2\chi_2+3/\gamma+1/\gamma^2)r_2 + \chi_1(3+2/\gamma)\sqrt{r_2}+\chi_1^2}{2r_2\sqrt{r_2}\left[(1+1/\gamma)\sqrt{r_2}+\chi_1\right]^3} + O\left(r_2^{\varepsilon-3}\right),$$

$$\left.\frac{\partial^2 \rho}{\partial r^2}\right|_{r_2,t} = \frac{1}{\gamma}\left.\frac{\partial^2 p}{\partial \rho^2}\right|_{r_2,t}, \qquad \left.\frac{\partial^2 u}{\partial r^2}\right|_{r_2,t} = \frac{1}{\sqrt{\gamma}}\left.\frac{\partial^2 p}{\partial r^2}\right|_{r_2,t}.$$

At the discontinuity we obtain

$$p(r) = 1 + \frac{c_0}{r^{1/4}} - \frac{1}{2}\frac{\chi_1}{1+1/\gamma}\frac{c_0}{r^{5/4}} + O\left(r^{\varepsilon-3/2}\right),$$

$$\rho(r) = 1 + \frac{c_0}{\gamma r^{3/4}} - \frac{1}{2\gamma}\frac{\chi_1}{1+1/\gamma}\frac{c_0}{r^{5/4}} + O\left(r^{\varepsilon-3/2}\right),$$

$$u(r) = \frac{c_0}{\sqrt{\gamma}\, r^{3/4}} - \frac{1}{2\sqrt{\gamma}}\frac{\chi_1}{1+1/\gamma}\frac{c_0}{r^{5/4}} + O\left(r^{\varepsilon-3/2}\right).$$

c_0 is determined from the initial conditions: if at $r = r_1$ (at the discontinuity) at the instant $t = t_0$ $p(r_1)$ is known (Fig. 1, $t = t_0$), then

$$c_0 = r_1^{3/4}\frac{p(r_1) - 1}{1 - \dfrac{1}{1+1/\gamma}\dfrac{\chi_1}{2\sqrt{r_1}}}.$$

It is noteworthy that in the cylindrical problem of the decay of a shock wave the third approximations immediately follow the linear approximations; the terms $\sim \ln r/r$ vanish.

Received
8 IV 1957

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.