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Abstract

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ON FINDING TIME CORRELATION FUNCTIONS FOR STATISTICAL SYSTEMS WITH LONG-RANGE INTERACTION AT TIMES UP TO THE FREE-PATH TIME

(Presented by Academician N. N. Bogolyubov, 19 IX 1956)

In a recent work ([1]) we derived a chain of equations for determining symmetric distribution functions:

$$\frac{\partial F_s(x_1t_1,\dots,x_{st}s)}{\partial t_k} = [H(x_k);\, F_s(x_1t_1,\dots,x_{st}s)]_{x_k} + \frac{1}{v}\int_\Omega [\Phi(|q_k - q|);\, F_{s+1}(x_1t_1,\dots,x_{st_{sxt}})]_{x_k}\, dx. \tag{1}$$

As the "initial" conditions we take

$$F_s(x_1t_1,\dots,x_{st}s) =$$

$$= \sum_{(m_1,\dots,m_l)} \frac{v^{s-l}}{l!}\frac{1}{m_1!\cdots m_l!}\sum_{(T)}\hat{T}\left\{\prod_{i=1}^{m_1}\delta(x_i - x_1)\prod_{i=m_1+1}^{m_1+m_2}\delta(x_i - x_{m_1+1})\cdots\right.$$

$$\left.\cdots\prod_{i=m_1+\dots+m_{l-1}+1}^{s}\delta(x_i - x_{m_1+\dots+m_{l-1}+1})\right\} F_l(tx_1, x_{m_1+1},\dots,x_{m_1+\dots+m_{l-1}+1}). \tag{2}$$

The functions $F_l(tx_1, x_{m_1+1},\dots,x_{m_1+\dots+m_{l-1}+1})$ that enter the right-hand side of (2) are the usual distribution functions of N. N. Bogolyubov ([2]). The summation $\sum_{(m_1,\dots,m_l)}$ is taken over all sets of integers $m_1,\dots,m_l$; the summation $\sum_{(T)}$ is taken over all $s!$ permutations of s symbols. In (1) and (2) we have assumed the usual limiting transition $N\to\infty$, $V\to\infty$, $v = V/N = \text{const}$.

To find the solution of (1), (2) in the case of a plasma in statistical equilibrium in the absence of external fields, we replace in (1) $H(x_k)$ by $T(p_k)$, and $\Phi(|q_k - q|)$ by $v\psi(|q_k - q|)$.

We shall seek the solution of (1), (2) in the form of a series in powers of the specific volume v:

$$F_s = F_s^{(0)} + vF_s^{(1)} + \dots; \tag{3}$$

in doing so, we use the expansion for the usual distribution functions obtained in ([2]) for the case of a plasma in statistical equilibrium in the absence of external fields.

It is easy to see that the solution of the zeroth approximation will be

$$F_s^{(0)}(x_1t_1, \dots, x_{st}s) = \prod_{(1\le i\le s)} f(p_i). \tag{4}$$

We shall seek the solution of the first approximation in the form

$$F_s^{(1)}(x_1t_1, \dots, x_{st}s) = \sum_{(1\le i<j\le s)} g(x_{it}i, x_{jt}j) \prod_{\substack{(1\le l\le s)\\(l\ne i,\ l\ne j)}} f(p_l), \tag{5}$$

where $g(x_{it}i, x_{jt}j)$ is a symmetric function with respect to interchange of its arguments; moreover, it is homogeneous in space and time, i.e.

$$g(x_1t_1, x_2t_2) = g(q_1 - q_2, t_1 - t_2, p_1, p_2). \tag{6}$$

We obtain

$$g(q, t, p_1, p_2) = g\left(q - \frac{p_1}{m}t, p_1, p_2\right) +$$

$$+ \int_0^t \int_\Omega K(q - q', t - t', p_1)\, g(-q', -t', p_2, p')\, dq'\, dp'\, dt', \tag{7}$$

where

$$q = q_1 - q_2, \qquad t = t_1 - t_2, \tag{8}$$

$$K(q, t, p) = -\sum_{(1\le\alpha\le3)} \frac{p^\alpha}{m\theta} \frac{e^{-\frac{p^2}{2m\theta}}}{(2\pi m\theta)^{3/2}} \frac{\partial\psi\left(\left|q - \frac{p}{m}t\right|\right)}{\partial q^\alpha}, \tag{9}$$

$$g(q, p_1, p_2) = g(q, 0, p_1, p_2). \tag{10}$$

Let us perform a Fourier transform with respect to the variables q:

$$g(q,t,p_1,p_2) = \frac{1}{(2\pi)^3}\int e^{ikq} r(k,t,p_1,p_2)\,dk; \tag{11}$$

then from (7) we obtain

$$r(k,t,p_1,p_2) = e^{-i\frac{kp_1}{m}t} g(|k|,p_1,p_2) +$$

$$+\int_0^t\int_\Omega K(k,t-t',p_1)\,r(-k,-t',p_2,p')\,dp'\,dt', \tag{12}$$

where

$$K(k,t,p) = -i\,\frac{kp}{m\theta}\,\frac{e^{-\frac{p^2}{2m\theta}-i\frac{kp}{m}t}}{(2\pi m\theta)^{3/2}}\,\nu(|k|); \tag{13}$$

$$\nu(|k|) = \int e^{-ikq}\psi(|q|)\,dq; \tag{14}$$

$$g(|k|,p_1,p_2) = f(p_1)\delta(p_1-p_2) + g(|k|)f(p_1)f(p_2); \tag{15}$$

$$g(|k|) = \int e^{-ikq} g(|q|)\,dq. \tag{16}$$

From the spatial and temporal homogeneity of $g(x_1t_1, x_2t_2)$,

$$r(k,t,p_1,p_2) = r(-k,-t,p_2,p_1). \tag{17}$$

Note that the problem of finding the solution of (12), (17) is equivalent to the problem of finding the solution

$$r(k,t,p_1,p_2) = e^{-i\frac{kp_1}{m}t} g(|k|,p_1,p_2) +$$

$$+\int_0^t\int K(k,t-t',p_1)\,r(k,t',p',p_2)\,dp'\,dt', \tag{18}$$

also satisfying condition (17). Equation (18) was obtained from (12) by replacing, in the integral term, $r(-k,-t',p_2,p')$ by $r(k,t',p',p_2)$. We shall now make use of a convenient separation of the problem (18), (17), namely, we can find the solution of equation (18) only in the region $t > 0$, and then use (17) to continue the solution obtained into the region $t < 0$. It is convenient to find the solution of equation (18) in the region $t > 0$ by the operational method. Let us introduce the following notation for Laplace transforms:

$$R(k, z, p_1, p_2) = \int_0^\infty r(k, t, p_1, p_2) e^{-zt}\, dt \qquad (\operatorname{Re} z > 0), \tag{19}$$

$$K(k, z, p_1) = \int_0^\infty K(k, t, p_1) e^{-zt}\, dt \qquad (\operatorname{Re} z > 0), \tag{20}$$

$$F(k, z, p_1, p_2) = \int_0^\infty e^{-i\frac{kp_1}{m}t - zt}\, g(|k|, p_1, p_2)\, dt \qquad (\operatorname{Re} z > 0). \tag{21}$$

For $R(k, z, p_1, p_2)$ we have

$$R(k, z, p_1, p_2) = F(k, z, p_1, p_2) + K(k, z, p_1)\, \frac{\int F(k, z, p_1, p_2)\, dp_1}{1 - \int K(k, z, p_1)\, dp_1}. \tag{22}$$

Using (13) and (15), after some calculations we obtain a rather cumbersome expression for $R(k, z, p_1, p_2)$, which, for lack of space, we do not give.

Performing the inverse Laplace transform, we obtain

$$r(k, t, p_1, p_2) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} R(k, z, p_1, p_2) e^{zt}\, dz \quad (t > 0), \tag{23}$$

$$r(k, t, p_1, p_2) = r(-k, -t, p_2, p_1) \quad (t < 0). \tag{24}$$

Fig. 1

Formulas (23), (24) give the solution in principle of the problem of finding the first approximation.

Let us say a few words about the analytic nature of $R(k, z, p_1, p_2)$. The only singularities of the function $R(k, z, p_1, p_2)$ at finite points of the z-plane are poles at the points $z = -ikp_1/m$, $z = -ikp_2/m$, and at the roots of the dispersion equation

$$1 + \frac{v(|k|)}{\theta} - \frac{v(|k|)}{\theta}\, z \int_0^\infty e^{-zt - \theta k^2 t^2/2m}\, dt = 0. \tag{25}$$

In the case $\lambda \gg 1$, $|k| \ll 1/r_D$, the roots of equation (25) are written approximately in the form (cf. (3))

$$z_1 \simeq i\omega_0 \left(1 + \frac{3}{2} r_D^2 k^2\right) - \sqrt{\frac{\pi}{8}}\,\omega_0\,\frac{1}{k^3 r_D^3} e^{-1/2 r_D^2 k^2}, \tag{26}$$

$$z_2 \simeq -i\omega_0 \left(1 + \frac{3}{2} r_D^2 k^2\right) - \sqrt{\frac{\pi}{8}}\,\omega_0\,\frac{1}{k^3 r_D^3} e^{-1/2 r_D^2 k^2}, \tag{27}$$

where $\omega_0 = \sqrt{4\pi e^2/vm}$ is the plasma frequency; $r_D = \sqrt{\theta v/4\pi e^2}$ is the Debye radius, e is the electron charge; $\theta = kT$. For large $|z|$, the function $R(k, z, p_1, p_2)$ behaves as follows: when z tends to infinity inside the shaded region (see Fig. 1), it tends to infinity, while when z tends to infinity outside this region, it tends ...

tends to zero. It is of interest, using the solution (23), (24), to obtain formulas for the function

$$g_1(q_1, t_1, q_2, t_2) = \iint g(x_1 t_1, x_2 t_2)\, dp_1 dp_2. \tag{28}$$

After some calculations we have

$$g_1(q_1, t_1, q_2, t_2) = \frac{1}{(2\pi)^3} \int e^{ikq} r_1(k, t)\, dk, \tag{29}$$

$$r_1(k, t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} R_1(k, z) e^{zt}\, dz \qquad (t > 0), \tag{30}$$

$$r_1(k, t) = r_1(-k, -t) \qquad (t < 0), \tag{31}$$

$$R_1(k, z) = \frac{(1 + g(|k|)) \displaystyle\int_0^\infty e^{-zt - \theta k^2 t^2/2m}\, dt}{1 + \dfrac{v(|k|)}{\theta} - \dfrac{v(|k|)}{\theta} z \displaystyle\int_0^\infty e^{-zt - \theta k^2 t^2/2m}\, dt}. \tag{32}$$

In order to obtain the asymptotic form of the integral (30) for large t, it is necessary, using the known method of asymptotic estimates for such integrals, to shift the path of integration in it to the left in such a way that the poles of the function $R_1(k, z)$ remain to the right of this path (see Fig. 2). Then

$$r_1(k, t) \simeq \operatorname*{res}_{z = z_1} R_1(k, z) e^{zt} + \operatorname*{res}_{z = z_2} R_1(k, z) e^{zt}. \tag{33}$$

Computing the residues, we obtain

Fig. 2

Figure 1: Fig. 2

$$r_1(k,t) \simeq (1 + g(|k|)) e^{-\gamma t} \cos \omega_0 t, \tag{34}$$

where γ is the decrement of Landau damping,

$$\gamma = \sqrt{\frac{\pi}{8}}\, \omega_0 \, \frac{1}{k^3 r_D^3} e^{-1/2k^2 r_D^2}.$$

Fig. 2

To obtain the behavior of $r_1(k,t)$ for small t, we return directly to the equation for $r_1(k,t)$:

$$r_1(k,t) = e^{-\theta k^2 t^2/2m}(1 + g(|k|)) - \frac{k^2 v(|k|)}{m} \int_0^t (t - t') e^{-\theta (t-t')^2 k^2/2m} r_1(k,t')\, dt', \tag{35}$$

which is obtained from (18) by integration over the momenta p_1, p_2. Replacing, for small t, the exponential factors in this equation by 1 and solving the resulting equation, we shall have

$$r_1(k,t) \simeq (1 + g(|k|)) \cos \omega_0 t. \tag{36}$$

Comparing (36) with (34), we conclude that, evidently, formula (34) gives a good representation of the behavior of the solution for all t.

It should be emphasized that, since we use the expansion (3) directly in powers of v, formula (34) is valid for times smaller than the mean free time.

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Note: Figure translations are in progress. See original paper for figures.