INEQUALITIES FOR DERIVATIVES ANALOGOUS TO S. N. BERNSTEIN' S INEQUALITY

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Abstract

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INEQUALITIES FOR DERIVATIVES ANALOGOUS TO S. N. BERNSTEIN' S INEQUALITY

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1. In the proposed article we consider certain classes of analytic functions in special domains (generally speaking, multiply connected) and establish exact inequalities for the derivatives of these functions on the boundary of the domain. Our inequalities also include S. N. Bernstein' s inequality ([1]) for entire transcendental functions of finite degree and some of its generalizations. The constructions set forth admit a rather considerable development, on which, however, we shall not dwell here. Likewise, we shall not dwell on how one can throw a bridge from our investigations to the well-known propositions of M. A. Lavrent' ev ([2]) on the change in the velocity of a flow at the boundary of a flow domain under variation of this boundary.

2. We shall denote by E a perfect point set on the real axis of the z-plane, having positive harmonic measure, and otherwise arbitrary, and by G the z-plane from which the set E has been removed. We shall agree further to call domains of type A, B, C the domains which are obtained respectively from the half-plane $\operatorname{Im}\zeta > 0$, the quadrant $\operatorname{Im}\zeta > 0,\ \operatorname{Re}\zeta > 0$ (or $\operatorname{Re}\zeta < 0$), and the half-strip $\operatorname{Im}\zeta > 0,\ \alpha < \operatorname{Re}\zeta < \beta$ $(-\infty < \alpha < \beta < \infty)$, by means of a finite or infinite number of finite rectilinear cuts beginning on the base of the domain and perpendicular to it, and capable of having limiting segments only on the boundary of the quadrant or half-strip. Here by the base of the domain we mean that part of its boundary which belongs to the real axis.

The starting point of our constructions is the following theorem.

Theorem 1. *The upper half-plane* $\operatorname{Im} z > 0$ *can always be mapped conformally onto some domain* Δ *of one of the types* A, B, C*, in such a way that the set* E *passes into the base of the domain* Δ*.*

For the proof of this theorem we consider several cases that may occur here and in each of them give a construction which leads to a definite mapping function together with the corresponding mapping domain Δ.

All possible continuations of the mapping function to the domain G by means of the principle of symmetry lead to a certain, generally speaking, many-valued in G function $\varphi(z)$. The imaginary part of this function $v(z) = \operatorname{Im}\varphi(z)$ in the domain G is single-valued and positive, and on the boundary of the domain G has at each point a limiting value equal to 0. By virtue of the very construction of the mapping function, the following alternative is obtained: for any positive $\delta < \pi/2$ there holds either the asymptotic equality

$$\frac{v(z)}{y} \to 1 \qquad (|z| \to \infty,\ \delta < \arg z < \pi - \delta), \tag{α}$$

or the asymptotic equality

$$\frac{v(z)}{y} \to 0 \quad (|z| \to \infty,\ \delta < \arg z < \pi - \delta). \qquad (\ )$$

To this one must add that, in case (α), the image domain Δ will necessarily be a domain of type A. From $\varphi(z)$ we construct the function $\omega(z) = e^{-i\varphi(z)}$, analytic in the domain G and, generally speaking, multivalued; moreover, the modulus of $\omega(z)$ in G is single-valued and greater than 1, while at every boundary point of G it is equal to 1.

3. We shall consider in G arbitrary analytic, generally speaking multivalued, functions. For each such function $f(z)$ we put

$$f^*(z) = \sup|f(z)|,$$

where the supremum is taken over all values of $|f(z)|$ at the point z. The finite or infinite quantity

$$\overline{\lim}_{|z|\to\infty,\ z\in G} \frac{\ln f^*(z)}{|z|}$$

will be called the degree of the function $f(z)$ in G, and we single out the aggregate of all functions of finite degree, to which, in particular, the function $\omega(z)$ belongs. The following theorem shows that $\omega(z)$ plays the role of a special majorant in the indicated aggregate.

Theorem 2. Let $f(z)$ be an analytic function of finite degree in G, satisfying two conditions:

a) at all points of the set E the function $f(z)$ has limiting values and they are bounded,

$$|f(x)| \leqslant 1 \qquad (x \in E);$$

b) to every $\varepsilon > 0$ there corresponds such a $\delta > 0$ that the ratio

$$f^*(z) : [\omega(z)]^{\sigma+\varepsilon}$$

for some $\sigma \geqslant 0$ tends to zero if $|z| \to \infty$, $\left|\arg z \pm \frac{\pi}{2}\right| \leqslant \delta$. In that case, at every point $z \in G$,

$$|f(z)| \leqslant |\omega(z)|^\sigma$$

and the equality sign in this relation at at least one point for at least one value of $f(z)$ is possible only when

$$f(z) \equiv e^{i\gamma}[\omega(z)]^\sigma,$$

where γ is a real constant.

4. Now we can define the classes, call them K_σ, of functions of which we spoke at the very beginning of the article. We shall say that the function $f(z)$ belongs to the class K_σ ($\sigma \geqslant 0$), if $f(z)$ is a function of finite degree in G, satisfying condition a) of Theorem 2, and if $f(z)$ can be represented in the form of a linear combination with complex coefficients of two functions, each of which satisfies condition b) of the theorem mentioned, and is real at the points E. For example, the class K_σ contains the function

$$c_1 \cos[\sigma\varphi(z)] + c_2 \sin[\sigma\varphi(z)],$$

if the complex numbers c_1, c_2 satisfy the inequality $|c_1| + |c_2| \leqslant 1$.

If case (α) of the alternative holds and $f(z)$ is an entire transcendental function of degree $\leqslant \sigma$, which on E is, in modulus, $\leqslant 1$, then it also belongs to the class K_σ.*

* It can be shown that the class K_0 consists only of constants.

Our inequality for the derivative is expressed by the following theorem.

Theorem 3. *Let $f(z)$ be a function of the class K_σ, where $\sigma > 0$, if the mapping domain Δ is a domain of type A or B, and $\sigma \geqslant \pi/\lambda$, if Δ is a domain of type C and λ is its width. In that case, at every point $x \in E$ where $f'(x)$ and $\omega'(x)$ exist, the inequality*

$$|f'(x)| \leqslant \sigma|\omega'(x)|,$$

holds, and it is sharp, as is shown by the function

$$f_0(z) = c_1[\omega(z)]^\sigma + c_2[\omega(z)]^{-\sigma} \quad (|c_1| + |c_2| = 1).$$

If the set E consists only of intervals, then the totality of extremal functions is exhausted by functions of the form $f_0(z)$.

5. The impetus for the present work was a recent article by Schaeffer ([3]), in which a closed set E on the real axis is considered, distributed "uniformly" in the sense that every interval of some fixed length contains a part of the set E whose measure is greater than some fixed positive number. Schaeffer proves that there exists a set $E^* \subset E$, with $\operatorname{mes}(E - E^*) = 0$, and a function $c(x)$ $(x \in E^*)$, such that, whatever the entire function $f(z)$ of degree $\leqslant \sigma$, the inequality

$$\sup_{x \in E} |f(x)| \leqslant 1$$

implies the inequality $|f'(x)| \leqslant \sigma c(x)$ at every point $x \in E^*$. As early as 1916, a similar type of result for trigonometric sums was established by I. I. Privalov ([4]). The results of A. Schaeffer and I. I. Privalov follow from our Theorem 3 in a sharpened form.

6. Applying the general Theorem 3 to sets E of particular form, we obtain a number of interesting special propositions, some of which we shall give. In doing so, omitting the derivations and formulations, we shall write out only the set E, the mapping function, the inequality, and the extremal subclass for each of the cases under consideration.

1°. E is the interval $[-1, 1]$;

$$\varphi(z) = i \ln\left(z + \sqrt{z^2 - 1}\right), \quad \lambda = \pi, \quad \sigma \geqslant 1;$$

$$|f'(x)| \leqslant \frac{\sigma}{\sqrt{1 - x^2}}; \quad f_0(z) = c_1 \left(z + \sqrt{z^2 - 1}\right)^\sigma + c_2 \left(z - \sqrt{z^2 - 1}\right)^\sigma$$

$$(|c_1| + |c_2| = 1).$$

The single-valued extremal functions are polynomials; they are obtained for integer $\sigma (= n)$ and have the form $c \cos n \arccos z$ $(|c| = 1)$.

2°. E is the half-axis $[0, \infty)$;

$$\varphi(z) = \sqrt{z}, \quad \sigma > 0;$$

$$|f'(x)| \leqslant \frac{\sigma}{2\sqrt{x}}; \quad f_0(z) = c_1 e^{i\sigma\sqrt{z}} + c_2 e^{-i\sigma\sqrt{z}} \quad (|c_1| + |c_2| = 1).$$

Entire functions of order $1/2$ belong to the extremal subclass for any $\sigma > 0$ and have the form $c \cos(\sigma\sqrt{z})$ $(|c| = 1)$.

3°. E consists of two intervals $(-\infty, -1]$, $[1, \infty)$;

$$\varphi(z) = \sqrt{z^2-1}, \quad \sigma > 0;$$

$$|f'(x)| \leqslant \sigma \left|\frac{x}{\sqrt{x^2-1}}\right|; \quad f_0(z) = c_1 e^{i\sigma\sqrt{z^2-1}} + c_2 e^{-i\sigma\sqrt{z^2-1}} \quad (|c_1| + |c_2| = 1).$$

Entire functions of finite degree belong to the extremal subclass for any $\sigma > 0$ and have the form $c\cos(\sigma\sqrt{z^2-1})$ $(|c| = 1)$.

4°. Using the result from 3° and certain special interpolation formulas, one can prove the following analogue of the inequality

A. A. Markov: if $\sigma \geq \sqrt{3}$ and $g(z)$ is an entire function of degree $\leq \sigma$, for which $\sup\limits_{x\in E} |g(x)| = 1$, where E is the same as in 3°, then $|g'(x)| \leq \sigma^2$ $(x \in E)$, and the equality sign is attained on the function $c\cos(\sigma\sqrt{z^2-1})$ at the points $x = \pm 1$, and only in this case.

5°. E consists of two intervals $[-1, \beta]$, $[\alpha, 1]$ $(-1 < \beta < \alpha < 1)$;

$$\varphi(z) = i\int_1^z \frac{(z-C)\,dz}{\sqrt{(z^2-1)(z-\alpha)(z-\beta)}}; \qquad |f'(x)| \leq \sigma \frac{|x-C|}{\sqrt{(1-x^2)(x-\alpha)(x-\beta)}};$$

the parameter C can be expressed in terms of elliptic functions by the formulas

$$k^2 = \frac{2(\alpha-\beta)}{(1+\alpha)(1-\beta)}; \qquad \alpha = 2\operatorname{sn}^2 a - 1 \quad (0 < a < K);$$

$$C = 1 - \sqrt{(1+\alpha)(1-\beta)}\,\frac{H'(a)}{H(a)}.$$

The single-valued extremal functions are polynomials; they are obtained if σ is an integer and $a = \frac{m}{\sigma}K$ $(m = 1, 2, \ldots, \sigma - 1)$, i.e., when a certain relation between the parameters α and β is satisfied.

6°. E consists of the intervals $(-\infty, -1]$, $[\beta, \alpha]$, $[1, \infty)$ $(-1 < \beta < \alpha < 1)$;

$$\varphi(z) = \int_1^z \frac{(z-\gamma)(z-\delta)}{\sqrt{(z^2-1)(z-\alpha)(z-\beta)}}\,dz; \qquad |f'(x)| \leq \sigma \left|\frac{(x-\gamma)(x-\delta)}{\sqrt{(x^2-1)(x-\alpha)(x-\beta)}}\right|.$$

The parameters γ and δ are obtained by means of the formulas

$$\gamma + \delta = \frac{\alpha + \beta}{2}; \qquad -\frac{(1-\gamma)(1-\delta)}{(1+\alpha)(1-\beta)} = \frac{\pi}{4KK'} + \frac{1}{2}\frac{d^2}{da^2}\ln H(a);$$

k and a are computed in the same way as in 5°. The single-valued extremal functions are integral functions of finite degree; they are obtained only when $\sigma = p, 2p, 3p, \ldots$, where

$$p = K' \frac{\operatorname{dn} a}{\operatorname{sn} a \operatorname{cn} a}.$$

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CITED LITERATURE

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4. I. I. Privalov, *The Cauchy Integral*, Saratov, 1916.

Note: Figure translations are in progress. See original paper for figures.