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Abstract

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ON BOUNDARY-VALUE PROBLEMS FOR EIGENVALUES AND EIGENFUNCTIONS OF ORDINARY DIFFERENTIAL EQUATIONS CONTAINING A SMALL PARAMETER AT THE HIGHEST DERIVATIVE

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Let two linear differential operators of even orders be given on some interval $[a, b]$:

$$L[u] = (-1)^n \sum_{i=0}^{2n} p_i(x)u^{(i)}(x); \qquad l[u] = (-1)^m \sum_{i=0}^{2m} q_i(x)u^{(i)}(x).$$

$$(m > 0,\ n - m = k > 0).$$

Consider the boundary-value problems

$$\text{I} \quad \begin{cases} \mu L[u] + l[u] = \lambda\rho(x)u; \\ u^{(s)}(a) = u^{(s)}(b) = 0 \quad (0 \le s \le n-1); \end{cases} \tag{1}$$

$$\text{II} \quad \begin{cases} l[v] = \lambda\rho(x)v; \\ v^{(s)}(a) = v^{(s)}(b) = 0 \quad (0 \le s \le m-1), \end{cases} \tag{2}$$

whose eigenvalues and normalized eigenfunctions we shall denote respectively by $\lambda_j(\mu), u_j(x, \mu)$ and $\lambda_j, v_j(x)$. We shall regard the parameter μ as small and positive; then, as was shown in paper ([1]), under certain restrictions the relations

$$\lambda_j(\mu) = \lambda_j + O(\mu^{1/2k}), \qquad u_j(x, \mu) = v_j(x) + O(\mu^{1/2k})$$

hold.

(We note that in paper ([1]) problems with more general boundary conditions than ours were considered.) The purpose of the present note is to compute the eigenvalues and eigenfunctions of problem I with accuracy up to and including terms of order $\mu^{1/2k}$.

1. We shall assume henceforth that the functions $p_i(x), q_i(x)$, and $\rho(x)$ satisfy the following conditions on the interval $[a,b]$: the functions $p_i(x)$ $(0 \le i \le 2n-2)$ are continuous; the functions $\rho(x)$ and $q_i(x)$ $(0 \le i \le 2m-2)$ are continuously differentiable $2k$ times; the functions $p_{2n-1}(x)$ and $q_{2m-1}(x)$ are $(2n-1)$ times differentiable, and the functions $p_{2n}(x)$ and $q_{2m}(x)$ are $2n$ times differentiable; moreover $p_{2n}(x) > 0$, $q_{2m}(x) > 0$. Under the assumptions made, the following lemma is valid:

Lemma 1. For every closed domain g of the plane of the complex variable λ, for sufficiently small positive values of μ $(0 < \mu \le \mu_0(g))$, equation (1) has $2n$ linearly independent solutions $\{u_i(x,\lambda,\mu)\}$ $(1 \le i \le 2n)$, which, together with their derivatives with respect to x up to order $(2n-1)$ inclusive, can be represented in the form

$$u_i^{(s)}(x,\lambda,\mu) = v_i^{(s)}(x,\lambda) + \mu^{1/k}\varphi_{is}(x,\lambda,\mu)$$

$$(1 \le i \le 2m,\ 0 \le s \le 2n-1), \tag{3}$$

$$u_i^{(s)}(x,\lambda,\mu) = \mu^{-s/2k}\left[\left(\varepsilon_{i-2m}^{2k}\frac{q_{2m}(x)}{p_{2n}(x)}\right)^{s/2k} h(x) + \mu^{1/2k}\varphi_{is}(x,\lambda,\mu)\right]\exp\left[\frac{\varepsilon_{i-2m}}{\mu^{1/2k}}\int_{x_i}^{x}\sqrt[2k]{\frac{q_{2m}(t)}{p_{2n}(t)}}\,dt\right] \tag{4}$$

$$(2m+1 \le i \le 2n, \quad 0 \le s \le 2n-1),$$

where $\{v_i(x,\lambda)\}$ $(1 \le i \le 2m)$ are linearly independent solutions of the degenerate equation (2), which are analytic functions of λ for $\lambda \in g$; $\{\varepsilon_\alpha\}$ are the roots of the equation $\varepsilon^{2k} + (-1)^k = 0$, and

$$\varepsilon_\alpha = e^{i\frac{\pi}{k}(\alpha - \frac{k+1}{2})} \quad (1 \le \alpha \le 2k);$$

$x_i = b$ $(2m+1 \le i \le 2m+k)$; $x_i = a$ $(2m+k+1 \le i \le 2n)$;

$$h(x) = \left(\frac{p_{2n}(x)}{q_{2m}(x)}\right)^{\frac{2n+2m-1}{4k}} \exp\int_a^x \left(\frac{q_{2m-1}(t)}{q_{2m}(t)} - \frac{p_{2n-1}(t)}{p_{2n}(t)}\right) dt;$$

$\varphi_{is}(x,\lambda,\mu)$ $(1 \le i \le 2n,\ 0 \le s \le 2n-1)$ are functions, $(2n-s)$ times continuously differentiable with respect to x on the interval $[a,b]$ and analytic with respect to λ and μ $(\lambda \in g,\ 0 < \mu \le \mu_0(g))$, and in the indicated domains of variation of the variables x, λ, and μ,

$$|\varphi_{is}(x, \lambda, \mu)| \leq \varphi_0,$$

$\varphi_0 = \varphi_0(g)$ is some constant.

2. The eigenvalues of the degenerate problem II are determined as the roots of the equation

$$F(\lambda) = \det \left\| \begin{matrix} v_i^{(s)}(a, \lambda) \\ v_i^{(s)}(b, \lambda) \end{matrix} \right\| = 0 \quad (1 \leq i \leq 2m,\ 0 \leq s \leq m-1), \tag{5}$$

where $\{v_i(x, \lambda)\}$ is a fundamental system of solutions of equation (2). We shall assume that these solutions are entire functions of λ.

Suppose that in some closed domain g one of the minors of order $(2n-1)$ of determinant (5), for example the minor $\Delta(\lambda)$, which is obtained by deleting the first column and the row with the quantities $v_i^{(s_0)}(a, \lambda)$ $(0 \leq s_0 \leq m-1)$, has no zeros; then the boundary-value problem

$$\text{II a} \quad \begin{cases} l[v] = \lambda \rho(x) v; \\ v^{(s)}(a, \lambda) = 0 \quad (0 \leq s \leq m-1,\ s \neq s_0), \quad v^{(s)}(b, \lambda) = 0 \quad (0 \leq s \leq m-1); \\ \displaystyle\int_a^b \rho(x) v^2(x, \lambda)\, dx = 1 \end{cases}$$

has in the given domain a solution $v_0(x, \lambda)$ unique up to a factor ± 1. This solution is an analytic function of λ for $\lambda \in g$ and, by virtue of the condition $\Delta(\lambda) \neq 0$, is linearly independent of the solutions of equation (2) $v_2(x, \lambda), \dots, v_{2m}(x, \lambda)$. We now turn to the analogous problem for equation (1):

$$\text{I a} \quad \begin{cases} \mu L[u] + l[u] = \lambda \rho(x) u; \\ u^{(s)}(a, \lambda, \mu) = 0 \quad (0 \leq s \leq n-1,\ s \neq s_0); \\ u^{(s)}(b, \lambda, \mu) = 0 \quad (0 \leq s \leq n-1); \\ \displaystyle\int_a^b \rho(x) u^2(x, \lambda, \mu)\, dx = 1. \end{cases}$$

Lemma 2. For sufficiently small values of μ $(0 < \mu \leqslant \mu_0(g))$, problem Ia has in the domain g a solution $u_0(x, \lambda, \mu)$, unique up to a factor ± 1. This solution depends analytically on λ and μ $(\lambda \in g,\ 0 < \mu \leqslant \mu_0(g))$ and can be represented in the form

$$\begin{aligned} u_0(x, \lambda, \mu) = c(\lambda, \mu)\{v_0(x, \lambda) + \mu^{1/2k} V_0(x, \lambda) + \mu^{1/k} V(x, \lambda, \mu) \\ + \mu^{m/2k}(W_1(x, \lambda, \mu) + W_2(x, \lambda, \mu))\}. \end{aligned} \tag{6}$$

Here $v_0(x,\lambda)$ is the solution of problem IIa;

$$V_0(x,\lambda)=A_k\left[-v_0^{(m)}(a,\lambda)\left(\frac{p_{2n}(a)}{q_{2m}(a)}\right)^{1/2k}\frac{\Delta_1(x,\lambda)}{\Delta(\lambda)}+v_0^{(m)}(b,\lambda)\left(\frac{p_{2n}(b)}{q_{2m}(b)}\right)^{1/2k}\frac{\Delta_2(x,\lambda)}{\Delta(\lambda)}\right],$$

where $\Delta_1(x,\lambda)$ and $\Delta_2(x,\lambda)$ are determinants obtained from the determinant $\Delta(\lambda)$: the first by replacing the row with the quantities $v_i^{(m-1)}(a,\lambda)$, the second –the row with the quantities $v_i^{(m-1)}(b,\lambda)$, by the functions $v_i(x,\lambda)$ $(2\leqslant i\leqslant 2m)$; in the case $s_0=m-1$, $\Delta_1(x,\lambda)\equiv 0$,

$$A_k=\sum_{\alpha=1}^{k}\cos\frac{\pi}{k}\left(\alpha-\frac{k+1}{2}\right);$$

$V(x,\lambda,\mu)$ is a function $2n$ times continuously differentiable with respect to x on the interval $[a,b]$ and analytic with respect to λ and μ $(\lambda\in g,\ 0<\mu\leqslant\mu_0(g))$, and in the indicated domains of variation of the variables x,λ,μ,

$$|V^{(s)}(x,\lambda,\mu)|\leqslant C\qquad(0\leqslant s\leqslant 2n-1),$$

$C=C(g)$ is some constant;

$$W_1(x,\lambda,\mu)=-\left(\frac{p_{2n}(b)}{q_{2m}(b)}\right)^{m/2k}\frac{v_0^{(m)}(b,\lambda)}{h(b)\delta_1}\begin{vmatrix}u_{2m+1}(x,\lambda,\mu)&\cdots&u_{2m+k}(x,\lambda,\mu)\\ \varepsilon_1^{m+1}&\cdots&\varepsilon_k^{m+1}\\ \cdots&\cdots&\cdots\\ \varepsilon_1^{n-1}&\cdots&\varepsilon_k^{n-1}\end{vmatrix};$$

$$W_2(x,\lambda,\mu)=-\left(\frac{p_{2n}(a)}{q_{2m}(a)}\right)^{m/2k}\frac{v_0^{(m)}(a,\lambda)}{h(a)\delta_2}\begin{vmatrix}u_{2m+k+1}(x,\lambda,\mu)&\cdots&u_{2n}(x,\lambda,\mu)\\ \varepsilon_{k+1}^{m+1}&\cdots&\varepsilon_{2k}^{m+1}\\ \cdots&\cdots&\cdots\\ \varepsilon_{k+1}^{n-1}&\cdots&\varepsilon_{2k}^{n-1}\end{vmatrix};$$

$\{u_i(x,\lambda,\mu)\}$ $(2m+1\leqslant i\leqslant 2n)$ are solutions of equation (1), representable in the form (4),

$$\delta_1=\begin{vmatrix}\varepsilon_1^m&\cdots&\varepsilon_k^m\\ \cdots&\cdots&\cdots\\ \varepsilon_1^{n-1}&\cdots&\varepsilon_k^{n-1}\end{vmatrix},\qquad \delta_2=\begin{vmatrix}\varepsilon_{k+1}^m&\cdots&\varepsilon_{2k}^m\\ \cdots&\cdots&\cdots\\ \varepsilon_{k+1}^{n-1}&\cdots&\varepsilon_{2k}^{n-1}\end{vmatrix};$$

$$c(\lambda, \mu) = 1 - \mu^{1/2k} \int_a^b \rho(x) v_0(x, \lambda) V_0(x, \lambda)\, dx.$$

3. With the aid of Lemma 2 one can prove the following theorems.

Theorem 1. *Suppose that in some open bounded domain G equation (5) has only simple roots. Then, for sufficiently small values of μ $(0 < \mu \le \mu_0(G))$, problems I and II have in this domain one and the same number of eigenvalues, and*

$$\lambda_j(\mu) = \lambda_j + \mu^{1/2k} \lambda_{j1}(\mu), \tag{7}$$

where $\lambda_{j1}(\mu)$ are functions analytic and bounded for sufficiently small values of μ $(0 < \mu \le \mu_{j0})$.

Remark. Theorem 1 establishes a one-to-one correspondence between the eigenvalues of problems I and II. In paper (1) such a correspondence was established only in one direction—from the eigenvalues of problem II to the eigenvalues of problem I.

Theorem 2. *If the operator $l[u]$ is self-adjoint, then under the assumptions of Theorem 1 the relations (7) take the form*

$$\lambda_j(\mu) = \lambda_j + \mu^{1/2k} c_k \left[\sqrt[2k]{p_{2n}(a) q_{2m}^{2k-1}(a) \left(v_j^{(m)}(a)\right)^2} + \sqrt[2k]{p_{2n}(b) q_{2m}^{2k-1}(b) \left(v_j^{(m)}(b)\right)^2} \right] + \mu^{1/k} \lambda_{j2}(\mu), \tag{8}$$

where $v_j(x)$ are the normalized eigenfunctions of problem II; $\lambda_{j2}(\mu)$ are functions analytic and bounded for sufficiently small μ $(0 < \mu \le \mu_{j0})$;

$$c_k = \sum_{\substack{\alpha_1 + \cdots + \alpha_k = k-1 \\ \alpha_1, \ldots, \alpha_k}} \varepsilon_1^{\alpha_1} \varepsilon_2^{\alpha_2} \cdots \varepsilon_k^{\alpha_k}.$$

In the case $n = 2,\ m = 1$ this formula becomes the formula obtained in paper (2).

Remark. At each point λ_j which is a simple root of equation (5), at least one minor of order $(2n - 1)$ of determinant (5), for example the minor $\Delta(\lambda)$, is nonzero. Thus, for the normalized eigenfunction of problem I $u_j(x, \mu)$, belonging to the eigenvalue $\lambda_j(\mu)$, we obtain

$$u_j(x, \mu) = u_0(x, \lambda_j(\mu), \mu), \tag{9}$$

where $u_0(x, \lambda, \mu)$ is the solution of problem Ia in a neighborhood of the point λ_j. By virtue of formula (6), for the function $u_0(x, \lambda, \mu)$ we shall have

$$u_j^{(s)}(x,\mu) = \left(1 - \mu^{1/2k}\int_a^b \rho(x)v_0(x,\lambda_j)V_0(x,\lambda_j)\,dx\right)\left[v_0^{(s)}(x,\lambda_j(\mu)) + \mu^{1/2k}V_0(x,\lambda_j) + O(\mu^{1/k})\right],$$

$$(0 \le s \le 2n-1, \quad a+\delta \le x \le b-\delta,\ \delta > 0), \tag{10}$$

since the functions $W_1(x,\lambda,\mu)$, $W_2(x,\lambda,\mu)$ and their derivatives with respect to x decrease exponentially as $\mu \to 0$ at any interior point of the interval $[a,b]$. For the eigenfunction itself and its first $(m-2)$ derivatives, formula (10) remains valid on the entire interval $[a,b]$. The terms $W_1(x,\lambda,\mu)$ and $W_2(x,\lambda,\mu)$ turn out to be significant only when calculating the higher derivatives of the eigenfunction: the first in a neighborhood of the point $x=b$, the second in a neighborhood of the point $x=a$.

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References

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Note: Figure translations are in progress. See original paper for figures.