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# ON THE SOLUTION OF THE BIHARMONIC PROBLEM

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**Abstract**

**Full Text**

**MATHEMATICS**

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**ON THE SOLUTION OF THE BIHARMONIC PROBLEM**

*(Presented by Academician S. L. Sobolev on 26 X 1956)*

1. Let  $r > 0$ ,  $M > 0$ , and  $r = \bar{r} + \alpha$ , where  $\alpha$  is an integer and  $0 < \alpha \leq 1$ . We shall say (as in [1]) that a measurable function  $f(\theta)$  of period  $2\pi$  belongs to the class  $H_2^{(r)}(M)$  if it: a) has a square summable over the period; b) has an absolutely continuous derivative of order  $(\bar{r} - 1)$ ; c) the derivative of order  $\bar{r}$ , which exists by virtue of b) almost everywhere, has a square summable over the period and satisfies, for all  $h$ , the condition

$$\left( \int_0^\pi |f^{(\bar{r})}(\theta + h) - f^{(\bar{r})}(\theta)|^2 d\theta \right)^{1/2} \leq M|h|^\alpha, \quad \text{if } 0 < \alpha < 1;$$

$$\left( \int_0^{2\pi} |f^{(\bar{r})}(\theta + h) - 2f^{(\bar{r})}(\theta) + f^{(\bar{r})}(\theta - h)|^2 d\theta \right)^{1/2} \leq M|h|, \quad \text{if } \alpha = 1.$$

2. Let  $\sigma$  be the open unit disk with center at the origin. We shall say that a function  $f(x, y)$ , measurable on  $\sigma$ , belongs to the class  $W_2^{(m)}(\sigma)$  if: a) it has on  $\sigma$  all generalized, in the sense of S. L. Sobolev [2], partial derivatives up to order  $m$  inclusive; b)  $f$  itself and all these derivatives have squares summable on  $\sigma$ , i.e.

$$D_0[f] \equiv \iint_\sigma f^2(x, y) dx dy < \infty,$$

$$D_j[f] = \iint_\sigma \sum_{\alpha+\beta=j} \frac{j!}{\alpha!\beta!} \left( \frac{\partial^j f}{\partial x^\alpha \partial y^\beta} \right)^2 dx dy < \infty \quad (j = 1, 2, \dots, m).$$

Let us note that, in passing to polar coordinates, the classical formula for change of variables in the integrals  $D_j[f]$  remains valid also in the case of generalized derivatives. This is easily proved by introducing the function  $f_h(x, y)$ , the mean of  $f(x, y)$  in the sense of S. L. Sobolev [2].

In what follows, instead of  $\lim_{\rho \rightarrow 1-0} F(\rho, \theta) = \varphi(\theta)$  almost everywhere, we shall write  $F_{\rho=1} = \varphi$ .

3. In this note the following theorems will be proved, established for  $m = 2$  in [3].

**Theorem 1.** Let  $u(\rho, \theta)$  be biharmonic in  $\sigma$ , belonging to the class  $W_2^{(m)}(\sigma)$  ( $m \geq 2$ );  $u|_{\rho=1} = \varphi(\theta)$ ;  $\frac{\partial u}{\partial \rho}\Big|_{\rho=1} = \psi(\theta)$ . Then

$$\varphi \in H_2^{(m-\frac{1}{2})}(M_1), \quad \psi(\theta) \in H_2^{(m-\frac{3}{2})}(M_2),$$

where  $M_1, M_2$  are completely determined constants.

**Theorem 2.** Let

$$\varphi(\theta) \in H_2^{(m-1)/2+\varepsilon_1}(M_1), \quad \psi(\theta) \in H_2^{(m-3)/2+\varepsilon_2}(M_2), \quad (1)$$

where  $0 \leq \varepsilon_1 < 1/2$ ,  $0 \leq \varepsilon_2 < 1/2$ . If  $u(\rho, \theta)$  is a biharmonic function in  $\sigma$  satisfying the boundary conditions

$$u|_{\rho=1} = \varphi, \quad \frac{\partial u}{\partial \rho}\Big|_{\rho=1} = \psi,$$

then

a) for  $\varepsilon_1 > 0$ ,  $\varepsilon_2 > 0$

$$u \in W_2^{(m)}(\sigma);$$

$$D_m[u] \leq \begin{cases} C_1 \left[ \frac{M_1}{\sqrt{\varepsilon_1}} + \frac{M_2 + \|\psi\|_{L_2}}{\sqrt{\varepsilon_2}} \right]^2, & m = 2, \\ C_2 \left[ \frac{M_1}{\sqrt{\varepsilon_1}} + \frac{M_2}{\sqrt{\varepsilon_2}} \right]^2, & m > 2, \end{cases} \quad (2)$$

where  $C_1, C_2$  are constants independent of  $\varepsilon_1, \varepsilon_2, M_1, M_2$ ;

b) for  $\varepsilon_1 = \varepsilon_2 = 0$ , among functions belonging to the classes  $H_2^{*(m-1)/2}$  and  $H_2^{*(m-3)/2}$ , but not belonging to  $H_2^{m'}$  and  $H_2^{m''}$ , where  $m' > m-1/2$ ,  $m'' > m-3/2$ , there exist both functions for which the corresponding integral  $D_m[u] < \infty$ , and functions for which  $D_m[u] = \infty$ .

4. Consider, for given  $\varepsilon_1 > 0$ ,  $\varepsilon_2 > 0$ , the classes of functions  $\varphi$  and  $\psi$  satisfying conditions (1). Denote by  $M_{\varepsilon_1}(\varphi)$  and  $M_{\varepsilon_2}(\psi)$  the corresponding least constants for which conditions (1) are satisfied. These classes form spaces of type (B), respectively with metrics  $\|\varphi\|_{\varepsilon_1}^* = M_{\varepsilon_1}(\varphi)$ ,  $\|\psi\|_{\varepsilon_2}^* = M_{\varepsilon_2}(\psi)$  (for  $m = 2$ ,  $\|\psi\|_{\varepsilon_2}^* = M_{\varepsilon_2}(\psi) + \|\psi\|_{L_2}$ ). Denote these spaces by  $H_{\varepsilon_1}$  and  $H_{\varepsilon_2}$ . Then Theorem 3 follows from Theorem 2.

**Theorem 3.** Let  $u(\rho, \theta)$  be a biharmonic function in  $\sigma$  satisfying the boundary conditions

$$u|_{\rho=1} = \varphi \in H_{\varepsilon_1}, \quad \frac{\partial u}{\partial \rho} \Big|_{\rho=1} = \psi \in H_{\varepsilon_2}.$$

Then

$$D_m[u] \leq C \left[ \frac{\|\varphi\|_{\varepsilon_1}^*}{\sqrt{\varepsilon_1}} + \frac{\|\psi\|_{\varepsilon_2}^*}{\sqrt{\varepsilon_2}} \right]^2, \quad (3)$$

where  $C$  is a positive constant independent of  $\varepsilon_1, \varepsilon_2$ .

Thus, the solution of the biharmonic problem is stable in the following sense: if  $\varphi$  and  $\psi$  are arbitrarily small in norm, respectively in the metrics of  $H_{\varepsilon_1}$  and  $H_{\varepsilon_2}$ , then for the biharmonic function  $u(\rho, \theta)$  in  $\sigma$  satisfying the boundary conditions

$$u|_{\rho=1} = \varphi, \quad \frac{\partial u}{\partial \rho} \Big|_{\rho=1} = \psi,$$

the integral  $D_m[u]$  will be arbitrarily small.

5. Theorems 1 and 2 can be obtained with the aid of certain auxiliary propositions, starting from Theorems 1 and 2 of S. M. Nikol'skii<sup>(4)</sup>. However, in this case the integral  $D_m[u]$  is estimated from above with another constant, in which the quantities  $\varepsilon_1$  and  $\varepsilon_2$  appearing in the denominator of (2) are not singled out. Theorem 1 can also be obtained by combining the theorems of S. L. Sobolev and V. I. Kondrashov<sup>(2)</sup>, if one takes into account that a function of the class  $W_2^{(m)}(\sigma)$  can be extended beyond  $\sigma$  with preservation of the class.

6. In what follows we shall need Lemmas 1 and 2.

**Lemma 1**<sup>(5)</sup>. Let the function  $u(x, y)$  have sufficiently many continuous partial derivatives in  $\sigma$ . If  $x + iy = \rho e^{i\theta}$ , then

$$\frac{\partial^n u}{\partial x^\alpha \partial y^\beta} = \sum_{j=1}^n \sum_{s+t=j} \frac{P_{\alpha,\beta}^{(s,t)}(\theta)}{\rho^{n-j+s}} \frac{\partial^j u}{\partial \rho^t \partial \theta^s}, \quad (4)$$

$$S_n[u] \equiv \sum_{\alpha+\beta=n} \frac{n!}{\alpha!\beta!} \left( \frac{\partial^n u}{\partial x^\alpha \partial y^\beta} \right)^2 \geq \left( \frac{\partial^n u}{\partial \rho^n} \right)^2. \quad (5)$$

where  $P_{\alpha,\beta}^{(s,t)}(\theta)$  are trigonometric polynomials of order  $\alpha + \beta = n$ , independent of  $u$ .

**Lemma 2.** Let the  $2\pi$ -periodic function

$$f(\theta) = \left( \frac{\gamma_0}{2} + \sum_{k=1}^{\infty} \gamma_k \cos k\theta + \eta_k \sin k\theta \right) \in H_2^{(n-\frac{1}{2}+\varepsilon)}(M),$$

where  $n \geq 1$ ,  $0 \leq \varepsilon < 1/2$ . Then: a) if  $\varepsilon > 0$ , then

$$\sigma_n(f) \equiv \sum_{k=1}^{\infty} k^{2n-1} (\gamma_k^2 + \eta_k^2) \leq \frac{\pi M^2}{2}; \quad (6)$$

b) if  $\varepsilon = 0$ , then among the boundary functions of the class  $H_2^{(n-\frac{1}{2})}$  there are both functions for which  $\sigma_n(f) < \infty$ , and functions for which  $\sigma_n(f) = \infty$  (for the definition of boundary functions see (6)); c) if  $\sigma_n(f) < \infty$ , then

$$f \in H_2^{(n-\frac{1}{2})} \left( \sqrt{2\pi\sigma_n(f)} \right). \quad (7)$$

**Proof.** a) By hypothesis

$$\left( \int_0^{2\pi} |f^{(n-1)}(\theta+h) - f^{(n-1)}(\theta)|^2 d\theta \right)^{1/2} \leq M|h|^{1/2+\varepsilon},$$

which is equivalent to the inequality

$$\sum_{k=1}^{\infty} k^{2(n-1)} (\gamma_k^2 + \eta_k^2) \sin^2 \frac{1}{2}kh \leq \frac{M^2}{4\pi} |h|^{1+2\varepsilon},$$

whence, by virtue of the inequality

$$\int_0^1 \left( \frac{\sin^2 \frac{1}{2}kh}{kh} \right)^2 dh \geq \frac{1}{k\pi^2}$$

(6) follows.

b) The functions

$$f_1(\theta) = \sum_{k=2}^{\infty} \frac{\cos(k\theta + (n-1)\pi/2)}{k^n \ln^{(1+\alpha)/2} k}, \quad f_2(\theta) = \sum_{\nu=1}^{\infty} \frac{\cos N^{2\nu}(\theta)}{N^{(2n-1)\nu}},$$

where  $0 < \alpha < 1/2$ ,  $N > 1$  is an integer, are boundary functions in the class  $H_2^{(n-\frac{1}{2})}$ , and for them  $\sigma_n(f_1) < \infty$ ,  $\sigma_n(f_2) = \infty$ .

c) follows from the identity

$$\left( \int_0^{2\pi} |f^{(n-1)}(\theta+h) - f^{(n-1)}(\theta)|^2 d\theta \right)^{1/2} = \left( 4\pi \sum_{k=1}^{\infty} k^{2(n-1)} (\gamma_k^2 + \eta_k^2) \sin^2 \frac{1}{2} kh \right)^{1/2}.$$

**7. Proof of Theorem 1.** Let  $u$  be a biharmonic in  $\sigma$  function belonging to  $W_2^{(m)}(\sigma)$ . As shown in note (3),

$$u(\rho, \theta) = \frac{(\rho^2 - 1)a_0 + 2a_0}{4} + \sum_{k=1}^{\infty} \rho^k \left\{ \left[ (\rho^2 - 1) \frac{\alpha_k - ka_k}{2} + a_k \right] \cos k\theta + \left[ (\rho^2 - 1) \frac{\beta_k - kb_k}{2} + b_k \right] \sin k\theta \right\}, \quad (8)$$

where  $a_k, b_k$  and  $\alpha_k, \beta_k$  are the Fourier coefficients of the functions  $\varphi = u|_{\rho=1}$  and  $\psi = u_\rho|_{\rho=1}$ .

From (8) we find that

$$\iint \left( \frac{\partial^m u}{\partial \rho^m} \right)^2 \rho d\rho d\theta \geq C_3 \sum_{k=m}^{\infty} k^{2m-1} (a_k^2 + b_k^2) + k^{2m-3} (\alpha_k^2 + \beta_k^2),$$

which, with the aid of (5) and (7), proves Theorem 1.

**Proof of Theorem 2.** a) Let  $0 < \varepsilon_1 < 1/2$ ,  $0 < \varepsilon_2 < 1/2$ . Then

$$\sum k^{2m-1} (a_k^2 + b_k^2) + k^{2m-3} (\alpha_k^2 + \beta_k^2) < \infty.$$

Put

$$u_\rho = \frac{(\rho^2 - 1)a_0 + 2a_0}{4} + \sum_{k=1}^{\mu} \rho^k \left\{ \left[ (\rho^2 - 1) \frac{a_k - ka_k}{2} + a_k \right] \cos k\theta + \left[ (\rho^2 - 1) \frac{\beta_k + kb_k}{2} + b_k \right] \sin k\theta \right\}.$$

Then

$$D_j[u_{\mu+\nu} - u_\mu] \leq d_j \sum_{\mu+1}^{\mu+\nu} k^{2m-1} (a_k^2 + b_k^2) + k^{2m-3} (\alpha_k^2 + \beta_k^2) \xrightarrow{\mu \rightarrow \infty} 0$$

uniformly with respect to  $\nu$  ( $j = 0, 1, \dots, m$ ).

By virtue of Lemma 4 ((1), p. 260), it follows from this that  $u \in W_2^{(m)}(\sigma)$ . Since

$$D_2[u] \leq C_4 \left( a_0^2 + \sum_{k=1}^{\infty} k^3 (a_k^2 + b_k^2) + k(\alpha_k^2 + \beta_k^2) \right),$$

$$D_m[u] \leq C_5 \sum_{k=m-2}^{\infty} k^{2m-1} (a_k^2 + b_k^2) + k^{2m-3} (\alpha_k^2 + \beta_k^2),$$

where  $C_4$  and  $C_5$  are positive constants independent of  $\varepsilon_1, \varepsilon_2, M_1, M_2$ , it follows, by virtue of (6), that (2) holds.

a) follows from part b) of Lemma 2.

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