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Abstract

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MATHEMATICS

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EXTENSIONS OF SPACES OF ABSTRACT FUNCTIONS CONNECTED WITH THE THEORY OF THE INTEGRAL

The theory of integration of abstract functions, studied by Bochner ([1]) and I. M. Gelfand ([2]), is conveniently constructed, following Bochner, by closing the integration operator

$$\int_{\Omega} \varphi(P)\, dP, \tag{1}$$

defined on the set $\mathfrak{M}$ of step functions $\varphi(P)$ with values in the Banach space X, i.e. for functions given by the equation

$$\varphi(P) = \xi_i, \quad P \in E_i \quad (i = 1, 2, \dots, N), \tag{2}$$

where ξ_i are certain elements of X; E_i are pairwise disjoint sets inside Ω, Lebesgue measurable.

An abstract function $\varphi(P)$ is called **measurable** if it serves as the limit of an almost everywhere convergent sequence of elements of $\mathfrak{M}$ (and hence, also of an enumerable set of such sequences). In addition to almost-everywhere convergence, we shall simultaneously consider another type of convergence, connected with a certain norm $\|\varphi\|_Y$ in the space of abstract functions, for which Bochner takes

$$\|\varphi\|_B = \int_{\Omega} \|\varphi(P)\|_X\, dP. \tag{3}$$

This $\|\varphi\|_Y$ must be chosen so that from the convergence of $\|\varphi_k\|$ to zero for $\mathfrak{M}$ it would follow that

$$\lim \int_{\Omega} \varphi_k(P)\, dP = 0. \tag{4}$$

In this case, for all sequences φ_k convergent in themselves in the sense of the norm $\| \; \|_Y$, there exists, and moreover uniquely, the limit of the integral $\int \varphi_k(P)\, dP$. A measurable function $\varphi(P)$ will be called **integrable in the sense of the norm** $\| \; \|_Y$, if for it there exists an almost everywhere convergent sequence $\varphi_k(P)$ from $\mathfrak{M}$, converging simultaneously in itself in this norm, i.e.

$$\|\varphi_k(P) - \varphi_l(P)\|_Y < \varepsilon \quad \text{for } k, l > N(\varepsilon). \tag{5}$$

Besides the Bochner norm, one can indicate other, broader norms that make it possible to construct a broader definition of the integral. Let, for example:

$$\|\varphi(\mathbf{P})\|_{\Phi_1} = \sup \frac{\|\int \omega(\mathbf{P})\varphi(\mathbf{P})\, dP\|_X}{\|\omega(\mathbf{P})\|_{L_\infty}}; \tag{6}$$

$$\|\varphi(\mathbf{P})\|_{\Phi_p} = \sup \frac{\|\int \omega(\mathbf{P})\varphi(\mathbf{P})\, dP\|_X}{\|\omega(\mathbf{P})\|_{L_{p'}}}, \qquad p > 1, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \tag{7}$$

where $\omega(\mathbf{P})$ ranges over the set of measurable real step functions of the point $\mathbf{P}$, and $\|\omega(\mathbf{P})\|_{L_\infty}$ denotes the least upper bound of the values assumed by $\omega(\mathbf{P})$ on a set of positive measure. (For $\| \; \|_{\Phi_1}$, in computing the sup it is enough to restrict oneself to such $\omega(\mathbf{P})$ as take two values ± 1.)

The norm $\| \; \|_{\Phi_p}$ is a natural generalization of the norm

$$\|\varphi(\mathbf{P})\|_{B_p} = \left[\int \|\varphi\|_X^p \, dP \right]^{1/p}, \tag{8}$$

which could have been considered instead of the Bochner norm. The norms $\| \; \|_{\Phi_1}$ or $\| \; \|_{\Phi_p}$ are broader than the Bochner norm. In the case when X is simply the real axis, $\| \; \|_{\Phi_1}$ coincides with this norm; however, it is easy to give examples where this is no longer so.

Example 1. Let $\varphi(x)$ be defined on the interval $0 \le x \le 1$ and take its values in l_2, the space of real sequences with convergent sum of squares. Denote by i_k the element whose k-th component is equal to one and all the others are equal to zero. Define the function $\varphi(x)$ by the formula $\varphi(x) = 2^k i_k / k$ on the interval $2^{-k} < x \le 2^{-k+1}$. Then $\varphi(x)$ is Bochner integrable.

Adjoining to $\mathfrak{M}$ all abstract functions of a point $\varphi(\mathbf{P})$ that serve as limits, in the norm $\| \; \|_{\Phi_p}$ and almost everywhere, of sequences $\varphi_k(\mathbf{P})$ does not turn this space into a complete one.

In order to obtain a space complete in the sense of $\| \; \|_{\Phi_p}$, unlike the Bochner case, it is necessary to complete $\mathfrak{M}$ by "ideal elements" –the limits of sequences converging in themselves in the sense of the norm $\| \; \|_{\Phi_p}$. We shall show how this is to be done. Corresponding to each point function integrable in the sense of $\| \; \|_{\Phi_p}$, we associate a certain additive abstract set function $\varphi(E)$, with values in our Banach space X, by the formula

$$\varphi(E) = \int_E \varphi(\mathbf{P})\,dP = \int_\Omega \eta_E(\mathbf{P})\varphi(\mathbf{P})\,dP, \tag{9}$$

where $\eta_E(\mathbf{P})$ is the characteristic function of the set E. Instead of $\varphi(\mathbf{P})$, we shall study these set functions $\varphi(E)$. In this way the integral of $\varphi(\mathbf{P})$ will correspond to $\varphi(\Omega)$. From this point of view, the very formulation of the question of the integral as an operator acting from the set $\mathfrak{M}$ into X loses its former meaning. Instead of studying extensions of such an operator, we shall study the internal properties of the new functional space we obtain.

Let Φ_p denote the space of all additive abstract functions of sets $\varphi(E)$, defined on all Lebesgue-measurable sets $E \subset \Omega$, with norm

$$\|\varphi(E)\|_{\Phi_p} = \sup \frac{\|\sum a_i \varphi(E_i)\|_X}{\|\omega(\mathbf{P})\|_{L_{p'}}}, \qquad p \geq 1, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

$$\omega(\mathbf{P}) = \alpha_i, \quad \mathbf{P} \in E. \tag{10}$$

Examples show that among the elements of Φ_p there are some that cannot be constructed by formula (8).

Example 2. Let Ω $(0 \leq x \leq 1)$; let X be l_2; $\psi_1(x) = i_1$; $\psi_2(x) = i_2$, $0 \leq x \leq 1/2$; $\psi_2(x) = i_3$, $1/2 < x \leq 1$; $\psi_3(x) = i_4$, $0 \leq x \leq 1/4$; $\psi_3(x) = i_5$, $1/4 < x \leq 2/4$; $\psi_3(x) = i_6$, $2/4 < x \leq 3/4$; $\psi_3(x) = i_7$, $3/4 < x \leq 1$, etc. Put

$$\psi_s(E) = \int_E \psi_s(P)\,dP.$$

The series $\sum_{s=1}^{\infty} \frac{2s^{1/2}}{s} \psi_s(E) = \varphi_0(E)$ converges in the norm $\| \; \|_{\Phi_1}$ and defines a function $\varphi_0(E)$ which is not the integral of any $\varphi_0(P)$.

The set Y_Ω of all measurable sets E such that $E \subset \Omega$ may be regarded as a metric space by taking as the distance between two elements E_1 and E_2 the measure of their symmetric difference

$$\rho(E_1, E_2) = m[(E_1 \setminus E_1 \cap E_2) \cup (E_2 \setminus E_1 \cap E_2)]. \tag{11}$$

We shall call a function $\varphi(E)$ **continuous** in Y_Ω if

$$\|\varphi(E_1) - \varphi(E_2)\|_X < \varepsilon \quad \text{when } \rho(E_1, E_2) < \delta(\varepsilon). \tag{12}$$

For functions $\varphi(E)$ that are integrals of point functions, this concept corresponds to absolute continuity. We introduce a more general concept, corresponding to the absolute continuity of integrals $\int |\varphi|^p \, d\Omega$, by means of the norm $\| \; \|_\Phi$.

Let $\| \; \|_\Phi$ be some norm in the space Φ of abstract set functions. To each $\varphi(E)$ one may put in correspondence a new function $\psi_\varphi(E_1)$ with values in Φ by the formula

$$\psi_\varphi(E_1) = \varphi(E \cap E_1). \tag{13}$$

We shall call a function $\varphi(E)$ **absolutely continuous in the norm** Φ if $\|\psi_\varphi(E_1)\|_\Phi < \varepsilon$ as soon as $mE_1 < \delta(\varepsilon)$. We shall call the norm $\| \; \|_\Phi$ **monotone** if $\|\varphi(E)\|_\Phi \geq \|\varphi(E \cap E_0)\|$ for all E_0. For a monotone norm, Theorem 1 holds.

Theorem 1. *The function $\psi_\varphi(E_1)$ corresponding to an absolutely continuous $\varphi(E)$ will be continuous in the space Y. Conversely, from the continuity of $\psi_\varphi(E_1)$ follows the absolute continuity of $\varphi(E)$.*

Using the classical Weierstrass theorems of the theory of continuous functions and applying them to $\psi_\varphi(E_1)$, we obtain Theorem 2.

Theorem 2. *The limit of a sequence of functions $\varphi_k(E)$, absolutely continuous in the norm Φ, converging in this norm, will be absolutely continuous.*

The norms $\| \; \|_{\Phi_p}$ will be monotone. Taking into account that the set $\mathfrak{M}$ consisted of continuous functions, we obtain the following consequence.

The closure Ψ_p of the set $\mathfrak{M}$ in $\| \; \|_{\Phi_p}$ consists of functions absolutely continuous in Φ_p.

A well-known example shows that Φ_p may also contain functions that are not absolutely continuous.

Example 3. Let $\varphi_\delta(E) = 1$ if $0 \in E$; $\varphi_\delta(E) = 0$ if $0 \notin E$. The function $\varphi_\delta(E)$ will have bounded norm, but will not be absolutely continuous.

For sets E, as consisting of vectors, one can define addition following Minkowski. Consider the norm of the difference $\varphi(E + Q) - \varphi(E)$, where Q is a vector. If

$$\|\varphi(E + Q) - \varphi(E)\|_\Phi < \varepsilon \quad \text{when } |Q| < \delta(\varepsilon), \tag{14}$$

then we shall say that the function $\varphi(E)$ is **continuous under translation**. We shall call the norm $\| \; \|_\Phi$ **homogeneous** if

$$\|\varphi(E + Q)\|_\Phi = \|\Phi(E)\|_\Phi. \tag{15}$$

To each function $\varphi(E)$ from Φ one can put in correspondence an abstract function of the vector Q, with values in Φ, by the formula

$$Z_\varphi(Q) = \varphi(E+Q). \tag{16}$$

Theorem 3. *If the function $\varphi(E)$ is continuous under translation in the homogeneous norm $\| \, \|_\Phi$, then $Z_\varphi(Q)$ is continuous, and conversely.*

In the same way as above, Theorem 4 is proved.

Theorem 4. *A sequence $\varphi_k(E)$ of elements continuous under translation and convergent in the homogeneous norm Φ has as its limit a function continuous under translation.*

It follows from this theorem that Ψ_p consists of functions continuous under translation. An example shows that in Φ_p, $p \geqslant 1$, there may exist absolutely continuous functions which are not continuous under translation.

Example 4. Define the functions $\chi_s(x)$, with values in the m-space of bounded sequences, by the equalities $\chi_1(x) = i_1,\ 0 \le x \le 1/2;\ \chi_1(x) = -i_1,\ 1/2 < x \le 1;\ \chi_2(x) = i_2,\ 0 \le x \le 1/4;\ \chi_2(x) = -i_2,\ 1/4 < x \le 2/4;\ \chi_2(x) = i_2,\ 2/4 < x \le 3/4;\ \chi_2(x) = -i_2,\ 3/4 < x \le 1$, and so on. Further, let

$$\chi_s(E) = \int_E \chi_s(x)\,dx, \qquad \chi_0(E) = \sum_{s=1}^{\infty} \chi_s(E)$$

(the convergence is weak, i.e. for every E, but not in norm!). For $p = 1$, $\chi_0(E)$ will be absolutely continuous in Φ_1, but not continuous under translation.

A special role among functions of sets $\varphi(E)$ is played by those which are representable in the form

$$\varphi(E) = \int_E \varphi(P)\,dP, \tag{17}$$

where $\varphi(P)$ are continuous abstract functions of the point P. It is not difficult to define the integral of discontinuous abstract functions. The set of such functions $\varphi(E)$ we shall denote by $\mathfrak{R}$.

Theorem 5. *Every function from $\mathfrak{R}$ serves as the limit, in the sense of any Φ_p, of a sequence of elements from $\mathfrak{R}$.*

Theorem 6. *The set of those elements of Φ_p which possess absolute continuity and continuity under translation coincides with the closure of the set $\mathfrak{R}$.*

Thus Theorem 6 singles out a closed subspace serving as the closure of $\mathfrak{R}$, and hence also $\mathfrak{M}$. It follows from it that Ψ_p is completely characterized by two properties: absolute continuity and continuity under translation. Theorem 6 is proved by constructing mean functions for the abstract functions $Z_\varphi(Q)$ generated by translation. These mean functions are continuous and give an approximation to $\varphi(E)$ by means of elements of $\mathfrak{R}$.

In the following note we shall show how embedding theorems for functional spaces, used in the theory of partial differential equations, can be transferred to abstract functions of sets.

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Note: Figure translations are in progress. See original paper for figures.