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MATHEMATICS

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Abstract

Full Text

MATHEMATICS

Ya. S. BUGROV

ON EMBEDDING THEOREMS

(Presented by Academician A. N. Kolmogorov on 11 IV 1957)

1. The present note is devoted to the study of questions connected with the converse embedding theorem of S. M. Nikol' skii.

In particular, in S. M. Nikol' skii' s theorem ((1), Theorem 3.52, p. 300) the following proposition is proved:

Let the function $\varphi(x_1, \dots, x_m)$, defined on the subspace $R_m = \{-\infty \le x_i \le \infty,\ i = 1, \dots, m\}$, $1 \le m \le n$, belong to the class $H_{pm}^{\left(r - \frac{n-m}{p}\right)}$ $(1 \le p \le \infty)$, where $r - \frac{n-m}{p} > 0$ and r is a positive number.

Then one can construct a function $f(x_1, \dots, x_n)$ of n variables having the properties:

a) $f \in H_{pn}^{(r)}$;

b) $f(x_1, \dots, x_m, 0, \dots, 0) = \varphi(x_1, \dots, x_m)$.

(For the definition of the class $H_{pn}^{(r)}$, see (1), p. 268.)

We shall consider the case when $r - \frac{n-m}{p} = 0$, and shall show that the theorem stated above remains valid if one assumes that $H_{pn}^{(0)} \equiv L_p(R_n)$. In addition, we investigate the question of the behavior of the derivatives of the extended function when approaching the manifold R_m; these results of ours strengthen the corresponding results of L. D. Kudryavtsev (2) in the same direction.

2. Theorem 1. Let $\varphi(x_1, \dots, x_m) \in L_p(R_m)$, $1 \le m \le n$, $1 \le p \le \infty$. Then one can construct a function $f(x_1, \dots, x_n)$ of n variables having the properties:

a) $f \in H_{pn}^{\left(\frac{n-m}{p}\right)}$;

b) $f(x_1, \dots, x_m, 0, \dots, 0) = \varphi(x_1, \dots, x_m)$.

The function f, which satisfies the conditions of the theorem, has the form

$$f(x_1, \dots, x_n) = \left(\frac{1}{2\pi}\right)^m \times$$

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$$\times \int_{R_m(t)} K(t_1, \dots, t_m, x_{m+1}, \dots, x_n) \varphi(t_1 + x_1, \dots, t_m + x_m)\, dt_1 \cdots dt_m, \quad (1)$$

where

$$K(t_1, \dots, t_m, x_{m+1}, \dots, x_n) =$$

$$= \int_{R_m(u)} \exp\left[-\sum_1^m (|u_k| + 1) \left(\sum_{m+1}^n x_j^2\right)^{1/2}\right] \exp\left[i \sum_1^m u_k t_k\right] du_1 \cdots du_m.$$

Remark 1. In the metric L_2, Theorem 1 can be strengthened.
If $\frac{n-m}{2}$ is an integer, then $f(x_1, \dots, x_n) \in W_2^{\left(\frac{n-m}{2}\right)}$. (For the definition of the class $W_p^{(l)}$, see (3).)

Theorem 2. *The function $f(x_1, \dots, x_n)$, defined by equality (1), has derivatives of arbitrary order (higher than $\bar{r}$), summable with weight*

$$\left\| \left(1 - \exp\left[-\left(\sum_{m+1}^n x_j^2\right)^{1/2}\right]\right)^{s + \bar{r} - \frac{n-m}{p} + \delta} f_{x_k}^{(r+s)}(x_1, \dots, x_n) \right\|_{L_p(R_n)} \le$$

$$\le C \|\varphi\|_{L_p(R_m)}, \quad (2)$$

where $s = 1, 2, \dots;$ $k = 1, 2, \dots, n;$ $1 \le p \le \infty,$ $\delta > 0;$ $\bar{r} = \frac{n-m}{p} - \alpha$ is an integer; $0 < \alpha \le 1$ (for $p = 2$ one can show that $\delta = 0$). The constant C depends on the quantities δ, p, m, n.

Theorems 1 and 2 admit further modifications in the sense that a function prescribed on the boundary of a domain can be extended in the required manner as a harmonic function. In particular, for the unit disk σ the following theorem is proved.

Theorem 1a. *Let a function $\varphi(\theta) \in L_p$ be prescribed on the boundary of the disk σ of radius $\rho = 1$.*

Then the function $u(\rho, \theta)$, harmonic inside this disk and such that $u(1, \theta) = \varphi(\theta)$, belongs to the class $H_p^{(1/p)}(\sigma)$ (here $m = 1,$ $n = 2,$ $\frac{n-m}{p} = \frac{1}{p},$ $1 \le p \le \infty$).

Proof. Let

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$$\varphi(\theta) \sim \sum_0^\infty (a_k \cos k\theta + b_k \sin k\theta) = \sum_0^\infty A_k(\theta)$$

be a function of period 2π, belonging to $L_p(0, 2\pi)$. Then, by Weierstrass' theorem, there exists a sequence of trigonometric polynomials $T_n(\theta)$ $(n = 1, 2, ...)$ such that

$$\|\varphi - T_n\|_p = \left(\int_0^{2\pi} |\varphi(\theta) - T_n(\theta)|^p \, d\theta \right)^{1/p} = C(n) \to 0 \quad \text{as } n \to \infty. \qquad (3)$$

Put

$$S_n = S_n(\varphi; \theta) = \sum_0^n A_k(\theta), \qquad \tau_n(\varphi; \theta) = \frac{S_n + \dots + S_{2n-1}}{n}.$$

It is easy to verify that

$$\tau_n(\varphi; \theta) = \frac{1}{\pi} \int_0^{2\pi} [2F_{2n-1}(u) - F_{n-1}(u)] \varphi(u + \theta) \, du,$$

where $F_k(u)$ is the Fejér kernel. Hence, by the generalized Minkowski inequality, taking into account that

$$\frac{1}{\pi} \int_0^{2\pi} F_k(u) \, du = 1,$$

$$\|\tau_n(\varphi; \theta)\|_p \le B \|\varphi\|_p \qquad (1 \le p \le \infty), \qquad (4)$$

where B is a constant.

Then from (3) and (4), and from the fact that $\tau_n(\varphi; \theta) = T_n(\theta)$ for all trigonometric polynomials of order n, we have

$$\|\varphi - \tau_n(\varphi)\|_p \leqslant (B + 1) C(n). \qquad (5)$$

Construct the harmonic function

$$u(\rho, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \sum_0^\infty \rho^k \cos kt \cdot \varphi(t + \theta) \, dt \qquad (0 \leqslant \rho < 1, \quad 0 \leqslant \theta \leqslant 2\pi)$$

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and the harmonic polynomial of order $(2n-1)$

$$\Phi_n(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\sum_0^\infty \rho^k \cos kt \cdot \tau_n(\varphi; t+\theta)\,dt.$$

Let us note that

$$\varphi(\theta) - \tau_n(\varphi;\theta) = \sum_1^{n-1}\frac{k}{n}A_{k+n}(\theta) + \sum_{2n}^\infty A_k(\theta),$$

i.e. it contains harmonics of order higher than n.

Applying Abel' s transformation for $0 \leqslant \rho < 1$, we obtain

$$\sum_0^\infty \rho^k \cos kt = \sum_0^\infty (k+1)\Delta^2\rho^k F_k(t),$$

where

$$\Delta^2\rho^k = \rho^k - 2\rho^{k+1} + \rho^{k+2} = \rho^k(1-\rho)^2 \geqslant 0.$$

Hence

$$u(\rho,\theta) - \Phi_n(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\sum_n^\infty (k+1)\Delta^2\rho^k F_k(t)\left[\varphi(t+\theta) - \tau_n(\varphi; t+\theta)\right]dt,$$

whence

$$\|u-\Phi_n\|_p^* = \left(\int_0^1\int_0^{2\pi}|u(\rho,\theta)-\Phi_n(\rho,\theta)|^p\rho\,d\rho\,d\theta\right)^{1/p} \leqslant$$

$$\leqslant \|\varphi - \tau_n(\varphi)\|_p\left(\int_0^1\left|\sum_n^\infty (k+1)\Delta^2\rho^k\right|^p d\rho\right)^{1/p} \leqslant \frac{C(B+1)C(n)}{n^{1/p}}, \qquad (6)$$

since

$$\int_0^1 \rho^{kp}(1-\rho)^{2p}\,d\rho = \frac{\Gamma(kp+1)\Gamma(2p+1)}{\Gamma(kp+2p+2)} < \frac{C}{k^{2p+1}}.$$

For the polynomials $\Phi_n(\rho,\theta)$ the inequalities (4) hold:

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$$\left\|\frac{\partial^l \Phi_n}{\partial \rho^l}\right\|_{L_p(R)} \leqslant C n^l \|\Phi_n\|_{L_p(R)};$$

$$\left\|\frac{\partial^l \Phi_n}{\partial \theta^l}\right\|_{L_p(R)} \leqslant C n^l \|\Phi_n\|_{L_p(R)} \qquad (l = 1, 2, \ldots), \tag{7}$$

where

$$\|\Phi_n\|_{L_p(R)} = \left(\int_R^1 \int_0^{2\pi} |\Phi_n(\rho, \theta)|^p \rho \, d\rho \, d\theta\right)^{1/p}, \qquad R > 0.$$

From (6) and (7), by the well-known Bernstein method for the inverse problem of approximation theory, we easily obtain

$$u(\rho, \theta) \in H_p^{(1/p)}(\sigma).$$

Remark 2. Condition (6) may be written in the form

$$\|u(\rho, \theta) - \Phi_n(\rho, \theta)\|_p^* = o(n^{-1/p}).$$

Hence, following Zygmund ([5]), one may assert that $u(\rho, \theta)$ is a smooth function belonging to the class $\mathrm{Lip}_p \frac{1}{p}$.

By definition $u(\rho, \theta) \in \mathrm{Lip}_p \, \alpha, \; 0 < \alpha \leq 1$, if the following properties hold:

$$\|u(\rho, \theta + h) - 2u(\rho, \theta) + u(\rho, \theta - h)\|_p^* = o(h^\alpha),$$

$$\left(\int_{1/2}^1 \int_0^{2\pi} |u(\rho + h, \theta) - 2u(\rho, \theta) + u(\rho - h, \theta)|^p \, \rho \, d\rho \, d\theta\right)^{1/p} = o(h^\alpha), \quad h \geq 0.$$

Theorem 2a. *Under the conditions of Theorem 1a, the harmonic function $u(\rho, \theta)$ has derivatives of all orders satisfying the condition*

$$\left\|\frac{\partial^l u}{\partial \rho^l}(1 - \rho)^{\, l - 1/p + \delta}\right\|_{L_p(R)} \leq C \|\varphi\|_p;$$

$$\left\|\frac{\partial^l u}{\partial \theta^l}(1 - \rho)^{\, l - 1/p + \delta}\right\|_{L_p(R)} \leq C \|\varphi\|_p,$$

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where $\delta > 0,\ l = 1, 2, ...$ *(for* $p = 2,\ \delta = 0$*);* C *is a constant.*

The **proof** of this theorem is based on the explicit representation of the function $u(\rho, \theta)$ and on the properties of the Poisson kernel.

We note that Theorem 1 can be proved for domains bounded by sufficiently smooth manifolds. Using Theorem 1, this assertion is proved in complete analogy with the proof of Theorem 6.2 of S. M. Nikol' skii (([1]), p. 317).

In conclusion I express my sincere gratitude to my scientific adviser S. M. Nikol' skii for his attention and assistance in the work.

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Note: Figure translations are in progress. See original paper for figures.

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