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P. I. KOVAL

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Abstract

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MATHEMATICS

P. I. KOVAL

ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF LINEAR DIFFERENCE AND DIFFERENTIAL EQUATIONS

(Presented by Academician S. L. Sobolev on 28 XII 1956)

The asymptotic behavior of solutions of linear difference equations with variable coefficients has been investigated by a number of authors (¹⁻³). The stability of solutions of systems of difference equations is the subject of works (⁴⁻⁶). The investigation of the asymptotic behavior of solutions of differential and difference equations is also treated in a number of sections of the monograph by I. M. Rapoport (⁷), where the author introduced the notion of an l -diagonal system of difference equations and an L -diagonal system of differential equations and studied these systems. Levinson (⁸), I. M. Rapoport (⁷), and T. G. Vas'kovskaya (⁹), in order to investigate the asymptotic behavior of solutions of a system of differential equations, reduced it to L -diagonal form. An analogous method was applied by I. M. Rapoport (⁷) in the theory of difference equations.

Below we present a method for reducing a system of linear difference (differential) equations to l -diagonal (L -diagonal) form, which makes it possible to cover a broader class of equations than in the investigations cited above. We examine in detail the application of this method to the investigation of the asymptotic behavior of solutions of linear difference and differential equations of second order.

1°. We shall consider the system of linear difference equations

$$x(s+1) = A(s)x(s) \quad (s = s_0, s_0 + 1, \dots), \quad (1)$$

whose matrix $A(s)$, beginning with a sufficiently large value of s , is nonsingular and has no multiple characteristic roots. We carry out in this system (1) the linear substitution

$$x(s) = B_0(s)[E + B_1(s) + B_2(s) + \dots + B_m(s)]y(s) \quad (2)$$

and try to choose the matrices $B_0, B_1, B_2, \dots, B_m$ so that the transformed system is l -diagonal:

$$y(s+1) = W(s)[E + C(s)]y(s), \quad (3)$$

where $W(s)$ is a diagonal matrix; $C(s) = \|c_{ij}(s)\|_1^n \in l$, i.e. $\sum_s^\infty |c_{ij}(s)| < \infty$ ($i, j = 1, 2, \dots, n$). By analogy with the method applied by T. G. Vas'kovskaya⁽⁹⁾, we define the matrices $B_0, B_1, B_2, \dots, B_m$ by the matrix equations

$$AB_0 = B_0W_0, \quad W_0B_1 = B_1W_0 + GW_0 + W_1,$$

$$\begin{aligned} W_0B_{p+1} = & B_{p+1}W_0 + GW_0B_p + \Delta B_pW_0 + B_{p-1}(W_1 + W_2 + \dots + W_{p-1}) + \\ & + [B_1(s+1) + B_2(s+1) + \dots + B_p(s+1)]W_p + W_{p+1} \end{aligned} \quad (4)$$

$$(p = 1, 2, \dots, m-1),$$

where

$$G(s) = E - B_0^{-1}(s+1)B_0(s), \quad \Delta B_p = B_p(s+1) - B_p(s);$$

$W_0(s)$,

$W_1(s), \dots, W_m(s)$ are diagonal matrices determined from the solvability conditions of these matrix equations. The diagonal elements of the matrices B_1, B_2, \dots, B_m remain arbitrary; we shall always assume them to be equal to zero.

Introduce the notation:

$$W^* = W_1 + W_2 + \dots + W_m, \quad W = W_0 + W^*,$$

$$B = B_1 + B_2 + \dots + B_m \quad \text{and} \quad H(s) = W^{-1}(s)B(s+1)W(s).$$

Theorem 1. *If the matrix $A(s)$, beginning with sufficiently large values of s , is nonsingular and has no multiple eigenvalues, and if there exists an m such that the matrices $B_0, B_1, \dots, B_m, W_0, W_1, \dots, W_m$, determined by equations (4), satisfy the conditions*

$$\lim_{s \rightarrow \infty} H(s) = 0, \quad W^{-1}GW_0B_m \in l, \quad W^{-1}\Delta B_mW_0 \in l,$$

$$W^{-1}(s)B_m(s+1)W^*(s) \in l, \quad W^{-1}(s)[B(s+1) - B_m(s+1)]W_m(s) \in l,$$

then the system (1) is reduced to the l -diagonal form (3) by the substitution (2).

Below we apply this theorem to the investigation of the asymptotic behavior of solutions of a second-order linear difference equation.

2°. The second-order linear difference equation

$$\Delta^2 z(s) = z(s+2) - 2z(s+1) + z(s) = a^2(s)z(s) \quad (5)$$

by means of the substitution $z = x_1$, $\Delta z = z(s+1) - z(s) = x_2$ is reduced to the system

$$x(s+1) = A(s)x(s), \quad A(s) = \begin{vmatrix} 1 & 1 \\ a^2(s) & 1 \end{vmatrix}. \quad (6)$$

This system is not l -diagonal. We shall show that in many cases, by means of the substitution (2), it can be reduced to the l -diagonal form (3). In the case under consideration we may put

$$B_0(s) = \beta(s) \begin{vmatrix} 1 & -1 \\ a(s) & a(s) \end{vmatrix}, \quad \beta(s) = \prod_{k=s_0}^{s-1} \{[a(k+1) + a(k)] : 2a(k+1)\}.$$

Theorem 2. *In order that system (6) be reducible to the l -diagonal form (3) by means of the substitution $x(s) = B_0(s)y(s)$, it is necessary and sufficient that $\Delta a(s) \in l$ and $a(\infty) \neq \{0, \pm 1\}$.*

The conditions of this theorem are satisfied, for example, by the function

$$a(s) = 2 + \frac{1}{s}.$$

In Theorems 3 and 4 the cases $a(\infty) = \{0, \infty\}$ are considered.

Theorem 3. *If $\lim_{s \rightarrow \infty} a(s) = \infty$ and there exists an m such that:*

1) *beginning with sufficiently large s ,*

$$\Delta^k a(s) \neq 0 \quad (k = 0, 1, 2, \dots, m), \quad \Delta^0 a = a;$$

2)

$$\Delta^k a(s) = o[\Delta^{k-1} a(s)] \quad (k = 1, 2, \dots, m+1); \quad (7)$$

3)

$$\left[\frac{\Delta^k a(s)}{a(s)} \right]^{\frac{m+1}{k}} \in l \quad (k = 1, 2, \dots, m+1),$$

then the substitution (2) reduces system (6) to the l -diagonal form (3).

Examples. The functions

$$a(s) = s^n, \quad \ln s, \quad \sum_{\sigma=1}^{s-1} \frac{1}{\sigma}$$

satisfy conditions (7) for $m = 1$. The function

$$a(s) = \prod_{\sigma=1}^{s-1} \left(1 + \frac{1}{\sqrt{\sigma}}\right)$$

does not satisfy the condi-

conditions (7) for $m = 1$, but satisfies these conditions for $m = 2$. The function

$$a(s) = e^{s^\sigma} \left(\frac{n-1}{n} < \sigma < \frac{n}{n+1}\right)$$

does not satisfy conditions (7) for $m = 1, 2, \dots, n-1$, but satisfies them for $m = n$. The function

$$a(s) = \prod_{\sigma=1}^{s-1} (1 + t_\sigma), \quad t_s = \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{9}, \dots, \frac{1}{n}, \frac{1}{n^2}, \dots$$

does not satisfy conditions (7) for any value $m = 1, 2, \dots$

Theorem 4. If $\lim_{s \rightarrow \infty} a(s) = 0$ and there exists an m such that:

- 1) beginning with sufficiently large s ,

$$\Delta^k a(s) \neq 0 \quad (k = 0, 1, 2, \dots, m),$$

$$\Delta^0 a = a;$$

- 2)

$$\Delta^k a(s) = o[a(s) \cdot \Delta^{k-1} a(s)] \quad (k = 1, 2, \dots, m+1); \quad (8)$$

- 3)

$$\frac{1}{a^m(s)} \left[\frac{\Delta^k a(s)}{a(s)} \right]^{\frac{m+1}{k}} \in l \quad (k = 1, 2, \dots, m+1),$$

then the substitution (2) brings system (6) to the l -diagonal form (3).

Examples. The functions

$$a(s) = \frac{1}{\ln s}, \quad \frac{1}{s^\sigma} \quad (0 < \sigma < 1), \quad \frac{(\ln s)^\sigma}{s} \quad (\sigma > 1)$$

satisfy conditions (8) for $m = 1$. The function

$$a(s) = \frac{(\ln s)^\sigma}{s}$$

$$\left(\frac{1}{n} < \sigma \leq \frac{1}{n-1}\right)$$

does not satisfy conditions (8) for $m = 1, 2, \dots, n-1$, but satisfies them for $m = n$. If the function $a = a(s)$ satisfies the conditions of one of Theorems 2-4, then

system (6) can be transformed to l -diagonal form, after which, for the solutions of the original difference equation (5), one can construct asymptotic expansions on the basis of the asymptotic formulas constructed by I. M. Rapoport (7) for solutions of l -diagonal systems of difference equations.

3°. The results stated above carry over to differential equations.

In the system of linear differential equations

$$x'(t) = A(t)x(t) \quad [\tau \leq t < \infty), \quad (9)$$

whose matrix, beginning with sufficiently large t , has no multiple eigenvalues, we make the substitution

$$x(t) = B_0(t)[E + B_1(t) + B_2(t) + \dots + B_m(t)]y(t) \quad (10)$$

and try to choose the matrices $B_0, B_1, B_2, \dots, B_m$ in such a way that the transformed system is L -diagonal:

$$y'(t) = [W(t) + C(t)]y(t), \quad (11)$$

where $W(t)$ is a diagonal matrix; $C(t) = \|c_{ij}(t)\|_1^n \in L[\tau, \infty)$, i.e.

$$\int_{\tau}^{\infty} |c_{ij}(t)| dt < \infty \quad (i, j = 1, 2, \dots, n).$$

To determine the matrices $B_0, B_1, B_2, \dots, B_m$, we construct, by analogy with difference equations, the matrix equations

$$\begin{aligned} AB_0 &= B_0W_0, & W_0B_1 &= B_1W_0 + G + W_1, \\ W_0B_{p+1} &= B_{p+1}W_0 + GB_p + B'_p + W_{p+1} + B_p(W_1 + W_2 + \dots + W_{p-1}) + \\ &+ (B_1 + B_2 + \dots + B_p)W_p \quad (p = 1, 2, \dots, m-1), \end{aligned} \quad (12)$$

where $G = B_0^{-1}B'_0$; $W_0(t), W_1(t), \dots, W_m(t)$ are diagonal matrices determined from the solvability conditions for these equations. The diagonal elements of the matrices B_1, B_2, \dots, B_m remain arbitrary; we shall assume them to be identically equal to zero.

Theorem 5. *If the matrix $A(t)$, starting from a sufficiently large value $t = \tau$, has no multiple eigenvalues, and if there exists an m for which the matrices $B_0, B_1, \dots, B_m, W_0, W_1, \dots, W_m$, determined by equations (12), satisfy the conditions*

$$\lim_{s \rightarrow \infty} (B_1 + B_2 + \dots + B_m) = 0, \quad GB_m \in L[\tau, \infty), \quad B_m \in L[\tau, \infty),$$

$$B_m(W_1 + W_2 + \dots + W_{m-1}) \in L[\tau, \infty), \quad (B_1 + B_2 + \dots + B_m)W_m \in L[\tau, \infty),$$

then system (9) is reduced to the L -diagonal form (11) by the substitution (10).

Below we apply this theorem to the study of the asymptotic behavior of solutions of a linear differential equation of the second order.

4°. The linear differential equation of the second order

$$z''(t) = a^2(t)z(t) \quad (\tau \leq t < \infty) \quad (13)$$

by means of the substitution $z = x_1$, $z' = x_2$ is reduced to the system

$$x'(t) = A(t)x(t), \quad A(t) = \begin{vmatrix} 0 & 1 \\ a^2(t) & 0 \end{vmatrix}. \quad (14)$$

Theorem 6. In order that system (14) be reducible to the L -diagonal form (11) by means of the substitution $x(t) = B_0(t)y(t)$, it is necessary and sufficient that $a'(t) \in L[\tau, \infty)$ and $a(\infty) \neq 0$.

In Theorem 7 the cases $a(\infty) = \{0, \infty\}$ are considered.

Theorem 7. If $a(t) \in L(\tau, t_1)$ for every finite value t_1 and there exists an m such that:

- 1) starting from a sufficiently large t , $a^{(k)}(t) \neq 0$ ($k = 0, 1, 2, \dots, m$), $a^{(0)} = a$;
- 2) $a^{(k)}(t) = o[a(t) \cdot a^{(k-1)}(t)]$ ($k = 1, 2, \dots, m + 1$);

(15)

$$3) \frac{1}{a^m(t)} \left[\frac{a^{(k)}(t)}{a(t)} \right]^{\frac{m+1}{k}} \in L[\tau, \infty) \quad (k = 1, 2, \dots, m + 1),$$

then the substitution (10) reduces system (14) to the L -diagonal form (11).

Examples. The functions $a(t) = e^t$, $\frac{1}{\ln t}$ satisfy conditions (15) for $m = 1$. The function

$$a(t) = \frac{(\ln t)^\sigma}{t} \left(\frac{1}{n} < \sigma < \frac{1}{n-1} \right)$$

does not satisfy conditions (15) for $m = 1, 2, \dots, n-1$, but satisfies these conditions for $m = n$.

If the function $a = a(t)$ satisfies the conditions of Theorem 6 or 7, then system (14) can be transformed to L -diagonal form, after which asymptotic expansions can also be constructed for the solutions of the original differential equation (13).

Kyiv State Pedagogical Institute
named after A. M. Gorky

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