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ON THE CONVERGENCE OF SERIES OBTAINED IN SOLVING NONLINEAR INTEGRAL EQUATIONS

(Presented by Academician A. N. Kolmogorov, 5 III 1957)

Below we consider the nonlinear integral equation

$$\varphi(x) = \int_0^1 K(x, y, \varphi(y), \lambda)\, dy, \tag{1}$$

where it is assumed that the function $K(x, y, \varphi, \lambda)$ is continuous in the aggregate of the variables and analytic in the last two. We assume that there is a representation

$$K(x, y, \varphi, \lambda) = \sum_{i,j=0}^{\infty} A_{ij}(x, y)\varphi^i \lambda^j, \tag{2}$$

where the functions $A_{ij}(x, y)$ are continuous, with $|A_{ij}(x, y)| < B_{ij}$, and the series

$$B(\varphi, \lambda) = \sum_{i,j=0}^{\infty} B_{ij}\varphi^i \lambda^j \tag{3}$$

converges for $|\varphi| < \rho_1$, $|\lambda| < \rho_2$. It is also assumed that $A_{00}(x, y) \equiv 0$.

One of the most widespread methods for solving equation (1) ([1,2]) consists in seeking this solution $\varphi(x)$ in the form of a series

$$\varphi(x) = \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \cdots \tag{4}$$

To determine the functions $\varphi_i(x)$, series (4) is substituted into equation (1), the right-hand side of this equation is expanded in a series in powers of λ, and then the coefficients of identical powers of λ in the left- and right-hand sides are equated. In this way one arrives at an infinite system of integral equations, from which the functions $\varphi_i(x)$ are determined successively. In order that series (4), with the formally constructed functions $\varphi_i(x)$, actually be a solution of equation (1), it is necessary to prove the uniform convergence of this series for values of λ from some interval. The proof of this uniform convergence is the main difficulty.

In works known to the author, the uniform convergence of series (4) follows from the existence of majorant series, i.e., of series $z = \lambda z_1 + \lambda^2 z_2 + \cdots$, convergent in some disk, with nonnegative coefficients z_i, such that $z_i \geq |\varphi_i(x)|$. The construction of a majorant series is usually carried out by a special device in each particular case. Thus, for example, in the book (2) a majorant series is constructed four times. In connection with the above, M. A. Krasnosel' skii proposed finding general conditions that would ensure the uniform convergence of a solution of equation (1) formally constructed in the form of series (4). Moreover, M. A. Krasnosel' skii put forward the hypothesis that, in the case of an integral equation of the form (1), formally constructed series always converge uniformly for values of λ from some interval.

In the present article some general propositions are given on the convergence of the formal solutions (4) of equation (1).

1°. The difficulty arises only in the case (2,3) when unity is an eigenvalue of the linear integral operator

$$A_{10}\varphi(x) = \int_0^1 A_{10}(x, y)\varphi(y)\, dy. \tag{5}$$

We assume that unity is a simple eigenvalue of the operator (5). In this case the equations for determining the functions $\varphi_i(x)$ have the form

$$\varphi_i(x) = \int_0^1 A_{10}(x, y)\varphi_i(y)\, dy + \int_0^1 M_i(x, y, \varphi_1(y), \dots, \varphi_{i-1}(y))\, dy, \tag{6}$$

where $M_i(x, y, \varphi_1, \dots, \varphi_{i-1})$ is a function whose explicit form is not difficult to indicate.

If for equation (1) it is possible to construct a formal solution in the form of the series (4), this means that each of equations (6) is solvable. Therefore the second terms on the right-hand sides of equations (6), for each i, must be orthogonal to the eigenfunction $q(x)$ of the kernel $A_{10}(y, x)$ corresponding to the eigenvalue equal to unity. When the orthogonality condition is fulfilled, the i-th equation (6) has a solution $\varphi_i(x)$ of the form

$$\varphi_i(x) = \varphi_i^0(x) + c_i p(x), \tag{7}$$

where $\varphi_i^0(x)$ is a known function; $p(x)$ is the eigenfunction of the kernel $A_{10}(x, y)$ corresponding to the eigenvalue equal to unity; c_i is an arbitrary constant.

If we now consider the $(i+1)$-st equation, then the second term on its right-hand side depends, generally speaking, on the constant c_i. In the principal cases the value of the constant is determined from the condition of orthogonality of this second term to the function $q(x)$. However, there may be cases when the second term is orthogonal to $q(x)$ for all values of c_i. Then the $(i + 1)$-st equation is solvable and its solution depends on a new arbitrary constant c_{i+1}. In this case, to determine the value of c_i, one uses the condition of orthogonality to $q(x)$ of the second term of the $(i + 2)$-nd equation. It may turn out that this orthogonality condition also holds identically with respect to c_i. Then one uses the next orthogonality condition, and so on.

In works known to the author, and in the examples considered by him, the process of determining the values of the constants c_i is established in a certain sense. In this connection we arrive at the following definition.

We shall say that the process of determining the functions $\varphi_i(x)$ stabilizes if it is possible to indicate such natural numbers m, l that the determination of the value of the arbitrary constant in formula (7), for $i \geq m$, is carried out from the condition of orthogonality of the second term on the right-hand side of the $(i + l)$-th equation to the function $q(x)$, i.e. from the condition

$$\int_0^1 \int_0^1 M_{i+l}[x, y, \varphi_1(y), \dots, \varphi_{i+l-1}(y)] q(x)\, dx\, dy = 0. \tag{8}$$

The cases considered in (^{2}) (pp. 37-55) and, for example, in (^{4}), correspond to a stabilizing process with $l = 1$. For the integral equation considered in (^{2}) (pp. 61-63), the process of determining φ_i also stabilizes, but $l = 2$.

Theorem 1. Let it be possible to construct formally a solution $\varphi(x)$ of equation (1) in the form of the series (4). Let the process of determining the functions $\varphi_i(x)$ stabilize.

Then the series (4) converges uniformly with respect to $x \in [0, 1]$, $|\lambda| \leqslant \rho$, where ρ is some positive number. The series (4) is a genuine solution of equation (1).

2°. The assertion of Theorem 1 follows from the possibility of constructing a universal majorant applicable to all stabilizing processes of determining the functions $\varphi_i(x)$.

From the condition of stabilization of the process it follows that the equation for determining the function φ_{m+l+1} has the form

$$\varphi_{m+l+1}(x) = \int_0^1 A_{10}(x, y) \varphi_{m+l-1}(y)\, dy + \int_0^1 G(x, y)\, dy +$$

$$+ c_{m+1} \int_0^1 H_{k+1}^l(x, y, \varphi_1(y), \dots, \varphi_{n_0}(y))\, dy, \tag{9}$$

where c_{m+1} is an arbitrary constant, to within which the function φ_{m+1} is determined; $G(x, y)$ is a function not depending on c_{m+1} (this function is determined in a sufficiently complicated way from the approximations found earlier).

The function H_{k+1}^l can be indicated; it has the fundamental property that

$$\int_0^1 \int_0^1 H_{k+1}^l(x, y, \varphi_1(y), \dots, \varphi_{n_0}(y)) q(x)\, dx\, dy = L \neq 0. \tag{10}$$

Consider the equation

$$z = RB(z, \lambda) + q_0 d, \tag{11}$$

where $R = 1 + \max |K(x, y)|$; $K(x, y)$ is the generalized resolvent of $A_{10}(x, y)$; $B(z, \lambda)$ is the function (3), in which $B_{10} = 0$, $q_0 = \max(|p(x)| + |q(x)|)$, $d = \lambda d_1 + \lambda^2 d_2 + \cdots$.

It is not hard to see that equation (11) has a solution of the form $z = \lambda z_1 + \lambda^2 z_2 + \cdots$.

The numbers z_i can be determined successively from equations obtained by comparing coefficients of like powers of λ. In this process one finds an $n_0 \geqslant m$ such that, for $n \geqslant n_0$, z_{n+l+1} has the form

$$z_{n+l+1} = R \left(\sum_{j=1}^{i} H^j(z_1, \dots, z_{n_0}) d_{n+l+1-j} + M^{n+l+1}(z_1, \dots, z_{n-1}) \right). \tag{12}$$

Let $k_i \geqslant |\varphi_i(x)|$ $(i = 1, 2, \dots, n_0 + l)$. Put $\widehat{H}^j = H^j(k_1, \dots, k_{n_0})$. By $P(\lambda)$ we denote such a polynomial of order $n_0 + l$ that coincides with the first terms of the expansion in powers of λ of the expression $k - RB(k, \lambda)$, where $k = \lambda k_1 + \lambda^2 k_2 + \cdots + \lambda^{n_0+l} k_{n_0+l}$. By $Q(\lambda)$ we denote a polynomial, also of order $n_0 + l$, coinciding with the first terms of the expansion in a series in powers of the function $B(k, \lambda)$.

Theorem 2. Let it be possible to construct formally a solution $\varphi(x)$ of equation (1) in the form of the series (4). Let the process of determining the functions $\varphi_i(x)$ stabilize.

Then the functions $\varphi_i(x)$ satisfy the inequalities

$$|\varphi_i(x)| \leqslant z_i \qquad (i = 1, 2, \dots), \tag{13}$$

where $z = \lambda z_1 + \lambda^2 z_2 + \cdots$ is a solution of the equation

$$\left(|z|\lambda^l + q_0 \sum_{j=1}^{i} \widetilde{H}\lambda^j\right)[z - RB(z,\lambda) - P(\lambda)] = q_0^2[B(z,\lambda) - Q(\lambda)]. \tag{14}$$

3°. Let us note that the assertions of Theorems 1 and 2 also apply to those cases in which the solution of equation (1) is sought in the form of series in fractional powers $\lambda^{1/n}$ of the parameter λ. In order to verify this, it suffices to introduce into consideration the new parameter $\mu = \lambda^{1/n}$.

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REFERENCES

1. A. I. Nekrasov, *An Exact Theory of Waves of Steady Type on the Surface of a Heavy Liquid*, 1951.
2. N. N. Nazarov, *Tr. Sredneaziatsk. Univ.*, no. 33, Tashkent (1941).
3. M. A. Krasnosel' skii, *Topological Methods in the Theory of Nonlinear Integral Equations*, 1956.
4. K. T. Akhmedov, *Tr. Inst. Fiziki i Matematiki AN AzerbSSR*, 7, mathematical series (1955).

Note: Figure translations are in progress. See original paper for figures.

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