# ASYMPTOTIC REPRESENTATIONS OF SPHEROIDAL FUNCTIONS WITH AZIMUTHAL INDEX \(m=1\)

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## Abstract

## Full Text

## MATHEMATICAL PHYSICS

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# ASYMPTOTIC REPRESENTATIONS OF SPHEROIDAL FUNCTIONS WITH AZIMUTHAL INDEX $m = 1$

*(Presented by Academician V. A. Fock, 20 II 1957)*

**1.** Let an equation of the form be given

$$Y'' + c^2 p(\eta) Y = 0, \qquad c \gg 1, \tag{1}$$

where the function $p(\eta)$ has $n$ poles of first order and zeros at the points $\eta_k$. Suppose, further, that it is possible to choose a "standard" [1] equation

$$y'' + P(\varphi) y = 0, \tag{2}$$

whose independent solutions $y_1(\varphi)$ and $y_2(\varphi)$ are known, and such that the poles and zeros $\varphi_k$ of the coefficient $P(\varphi)$ can be put into a one-to-one and monotone correspondence with $\eta_k$ so that poles correspond to poles, and zeros to zeros of the same order. Then the asymptotic representation of the general solution of equation (1) has the form ($B_1$ and $B_2$ are arbitrary constants)

$$Y(\eta) = \sqrt[4]{\frac{P[\varphi(\eta)]}{p(\eta)}} \{B_1 y_1[\varphi(\eta)] + B_2 y_2[\varphi(\eta)]\}, \tag{3}$$

where the relation between the independent variables $\varphi = \varphi(\eta)$ is defined by

$$\int_{\varphi_k}^{\varphi} \sqrt{P(\varphi)}\, d\varphi = c \int_{\eta_k}^{\eta} \sqrt{p(\eta)}\, d\eta \tag{4}$$

and by the additional conditions

$$\int_{\varphi_k}^{\varphi_i} \sqrt{P(\varphi)}\, d\varphi = c \int_{\eta_k}^{\eta_i} \sqrt{p(\eta)}\, d\eta \qquad (i = 1, 2, \dots, k-1, k+1, \dots n). \tag{5}$$

The function $\varphi = \varphi(\eta)$ transforms the corresponding $\eta_k$ and $\varphi_k$ into one another, with $\varphi'(\eta) \neq 0$, $\varphi'(\eta) \neq \infty$. Conditions (5) can be satisfied if the coefficient $P(\varphi)$ contains $n-1$ free parameters (cf. [1,2]).

**2.** The function $Y(\eta)$, connected with the angular spheroidal function $S^{(1)}_{1,l}(c,\eta)$ [3] by the relation

$$Y(\eta) = \sqrt{1-\eta^2}\, S^{(1)}_{1,l}(c,\eta), \tag{6}$$

satisfies equation (1) with the coefficient of $Y$

$$p(\eta) = 1 + \frac{\beta}{1-\eta^2}, \qquad \beta = \chi - 1 + \frac{2}{c^2}, \tag{7}$$

where $\chi$ is connected with the separation constant $A$, introduced in [3], by the formula

$$A = -c^2\chi - 2. \tag{8}$$

The function $Y(\eta)$ is finite in the interval $(-1,1)$ and vanishes at its endpoints.

Since equation (1) and the boundary conditions are symmetric with respect to $\eta = 0$, the eigenfunctions of the equation are even or odd, and one may restrict consideration to the interval $(0,1)$ with the boundary conditions (cf. (1))

$$\begin{aligned} &Y(1) = 0, \qquad Y'(0) = 0 \quad \text{(for even } Y\text{)},\\ &Y(1) = 0, \qquad Y(0) = 0 \quad \text{(for odd } Y\text{)}. \end{aligned} \tag{9}$$

The function (7) has a pole of first order at $\eta = 1$ and a zero at the point $\eta_1 = +\sqrt{1+\beta}$, real for $\beta \geq -1$. For $\beta < 0$ the zero $\eta_1$ lies inside the interval $(0,1)$, for $\beta > 0$ outside this interval, and for $\beta = 0$ it merges with the pole, and the function $p(\eta)$ becomes a constant.

3. As the comparison equation we take equation (2) with coefficient, for $y$,

$$P(\varphi) = 1 + \frac{b}{\varphi}, \tag{10}$$

which takes into account all the singularities of the function $p(\eta)$. The solutions of this equation are degenerate hypergeometric functions with indices $k = -i\,\frac{b}{2}$, $m = \frac{1}{2}$ and argument $2i\varphi$ (see, for example, (4), Chap. 16). The relation $\varphi = \varphi(\eta)$ is specified by

$$\int_{\varphi_1}^{\varphi} \sqrt{P(\varphi)}\, d\varphi = c \int_{\eta}^{\eta_1} \sqrt{P(\eta)}\, d\eta \tag{11}$$

under the additional condition (5), which puts the poles of the functions $p(\eta)$ and $P(\varphi)$ into correspondence and makes it possible to relate the parameters $b$ and $\beta = -(1-\eta_1^2)$.

The boundary condition at $\eta = 1$ is satisfied by the solution

$$Y(\eta) = B\sqrt[4]{\frac{P(\varphi)}{p(\eta)}}\, M_{-i\frac{b}{2},\,\frac{1}{2}}(2i\varphi). \tag{12}$$

Using the asymptotic representation by the method of B. W. K.,

$$M_{-i\frac{b}{2},\,\frac{1}{2}}(2i\varphi) = \tag{13}$$

$$= \frac{2ie^{3\pi b/4}}{\sqrt[4]{P(\varphi)}}\sqrt{\frac{\operatorname{sh}(\pi b/2)}{\pi b/2}}\,\sin\left(\int_{\varphi_1}^{\varphi}\sqrt{P(\varphi)}\,d\varphi - \frac{b}{2} + \frac{b}{2}\ln\left(-\frac{b}{2}\right) - \operatorname{arc}\Gamma\left(1+i\frac{b}{2}\right)\right),$$

valid for large $\varphi$, but arbitrary $b$, we obtain from the boundary condition at $\eta = 0$ formulas for determining the eigenvalues $\beta$:

$$c\int_0^{\eta_1}\sqrt{p(\eta)}\,d\eta = \left(l+\frac{1}{2}\right)\frac{\pi}{2} - \operatorname{arc}\chi^-(b) \qquad (b<0,\ -1<\beta\le 0),$$

$$c\int_0^{1}\sqrt{p(\eta)}\,d\eta = \left(l+\frac{3}{2}\right)\frac{\pi}{2} - \operatorname{arc}\chi^+(b) \qquad (b>0,\ 0\le\beta<\infty). \tag{14}$$

Here the functions $\chi^-(b)$ and $\chi^+(b)$ are defined as the ratio of the asymptotic representation of $\Gamma\left(1+i\frac{b}{2}\right)$ for $b<0$ and $b>0$, respectively, to $\Gamma\left(1+i\frac{b}{2}\right)$ itself (cf. (5), pp. 568-569).

$$\chi^-(b) = \frac{\sqrt{-\pi b}\exp\left\{\frac{b\pi}{4} + i\left[\frac{b}{2}\ln\left(-\frac{b}{2}\right) - \frac{b}{2} - \frac{\pi}{4}\right]\right\}}{\Gamma\left(1+i\frac{b}{2}\right)} \qquad (b<0),$$

$$\chi^+(b) = \frac{\sqrt{\pi b}\exp\left\{-\frac{b\pi}{4} + i\left[\frac{b}{2}\ln\frac{b}{2} - \frac{b}{2} + \frac{\pi}{4}\right]\right\}}{\Gamma\left(1+i\frac{b}{2}\right)} \qquad (b>0). \tag{15}$$

As $\beta \to 0$ $(b \to 0)$, both relations (14) pass into the exact formula

$$c = (l+1)\frac{\pi}{2} \qquad (\beta = 0), \tag{16}$$

and the function (12) into the exact solution of equation (1) for $\beta = 0$. As $b \to -\infty$ $(\beta < 0)$ and as $b \to \infty$ $(\beta > 0)$, relations (14) take the form

$$c\int_0^{\eta_1}\sqrt{p(\eta)}\,d\eta=\left(l+\frac{1}{2}\right)\frac{\pi}{2}\qquad(b\to-\infty,\ \beta<0);\tag{17}$$

$$c\int_0^{1}\sqrt{p(\eta)}\,d\eta=\left(l+\frac{3}{2}\right)\frac{\pi}{2}\qquad(b\to\infty,\ \beta>0).\tag{18}$$

The first expression can be obtained by taking into account, with the aid of the Airy equation,

$$y''-\varphi y=0\tag{19}$$

only the root $\eta_1$ for $\eta_1<1$, and the second with the aid of the equation

$$y''+\frac{1}{\varphi}y=0,\qquad y=\sqrt{\varphi}\,J_1(2\sqrt{\varphi})\tag{20}$$

**Table 1**

Values of $\kappa_l=\beta_l+1-\dfrac{2}{c^2}$

| $l$ | $c=3$: exact values | $c=3$: by formula (14) | $c=3$: by Airy functions (above the line) and by Bessel functions (below the line) | $c=5$: exact values | $c=5$: by formula (14) | $c=5$: by Airy functions (above the line) and by Bessel functions (below the line) | $c=7$: exact values | $c=7$: by formula (14) | $c=7$: by Airy functions (above the line) and by Bessel functions (below the line) |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0,1672 | 0,166 | 0,097 | 0,1340 | 0,1331 | 0,115 | 0,1084 | 0,1079 | 0,099 |
| 1 | 0,8395 | 0,8397 | 0,977 | 0,5057 | 0,5054 | 0,471 | 0,3759 | 0,3757 | 0,363 |
| 2 | 1,5820 | 1,584 | 1,581 | 0,8559 | 0,8560 | 0,771 | 0,6232 | 0,6231 | 0,602 |
| 3 | 2,4898 | 2,493 | 2,508 | 1,2169 | 1,2171 | 1,251 | 0,8549 | 0,8550 | 0,810 |
| 4 | 3,6060 | 3,610 | 3,568 | 1,6263 | 1,6268 | 1,626 | 1,0857 | 1,0858 | 1,102 |
| 5 | 4,9413 | 4,947 | 4,945 | 2,1069 | 2,1076 | 2,116 | 1,3356 | 1,3358 | 1,347 |

| $l$ | $c = 3$: exact values | $c = 3$: by formula (14) | $c = 3$: by Airy functions (above the line) and by Bessel functions (below the line) | $c = 5$: exact values | $c = 5$: by formula (14) | $c = 5$: by Airy functions (above the line) and by Bessel functions (below the line) | $c = 7$: exact values | $c = 7$: by formula (14) | $c = 7$: by Airy functions (above the line) and by Bessel functions (below the line) |
|---|---|---|---|---|---|---|---|---|---|
| 6 | 6,4978 | 6,504 | 6,454 | 2,6659 | 2,6670 | 2,655 | 1,6194 | 1,6196 | 1,618 |
| 7 | 8,2762 | 8,284 | 8,276 | 3,3049 | 3,3062 | 3,319 | 1,9430 | 1,9433 | 1,948 |
| 8 | 10,2766 | 10,285 | 10,230 | 4,0240 | 4,0257 | 4,010 | 2,3078 | 2,3082 | 2,303 |
| 9 | | | | 4,8234 | 4,8252 | 4,825 | 2,7140 | 2,7146 | 2,717 |
| 10 | | | | | | | 3,1614 | 3,1621 | 3,155 |
| 11 | | | | | | | 3,6500 | 3,6507 | 3,652 |
| 12 | | | | | | | 4,1796 | 4,1805 | 4,173 |
| 13 | | | | | | | 4,7507 | 4,7513 | 4,7511 |

($J_1$ is the Bessel function), taking into account for $\eta_1 > 1$ only the pole of the function $p(\eta)$, if the zeros of the functions $v(\varphi)$, $v'(\varphi)$, $J_1(\varphi)$, $J_1'(\varphi)$ are replaced by their approximate values[6].

As $\beta \to 0$, formulas (17) and (18) cease to be valid, i.e., representations by means of Airy and Bessel functions are valid only when the root $\eta_1$ is sufficiently far from the pole $\eta = 1$.

The eigenvalues $\varkappa_l(c)$ in Table 1 were computed: 1) by the exact formulas; 2) by approximate relations with the aid of the expressed hypergeometric functions (14); 3) by Airy functions (17); 4) by Bessel functions. In the last case the roots of $J_1(\varphi)$ and $J_1'(\varphi)$ were not replaced by their approximate values, which gives better results. For Airy functions, formula (17) is better than the formula with exact roots $v(\varphi)$ and $v'(\varphi)$.

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*Note: Figure translations are in progress. See original paper for figures.*