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Abstract

Full Text

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ON ONE METHOD OF NUMERICAL INTEGRATION OF DIFFUSION EQUATIONS

(Presented by Academician S. L. Sobolev on 23 III 1957)

We shall solve the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad 0 < t \leq T, \tag{1}$$

with initial conditions

$$u(x, 0) = f(x), \qquad 0 \leq x \leq 1, \tag{2}$$

and boundary conditions

$$u(0, t) = u(1, t) = 0, \qquad t > 0. \tag{3}$$

Introduce the notation: $h^2/l = \omega = \text{const}$; $m = \left[\frac{T}{l}\right]$, $h = \frac{1}{n}$ (n an integer); h and l are the steps with respect to x and t, respectively; $\varphi_{ik} = \varphi(ih, kl)$; $\Delta\varphi_{ik} = (\varphi_{i,k+1} - \varphi_{i,k})/l$; $\Delta^2\varphi_{ik} = (\varphi_{i-1,k} - 2\varphi_{i,k} + \varphi_{i+1,k})/h^2$; $\|\varphi^{(k)}\| = \max_i |\varphi_{i,k}|$; $\|A\| = \|a_{ij}\| = \max_i \sum_{j=1}^{n-1} a_{ij}$; $\varphi^{(k)} = \{\varphi_{1,k}, \varphi_{2,k}, \dots, \varphi_{n-1,k}\}$ is an $(n-1)$-dimensional column vector.

Let us recall two well-known, fundamentally different numerical methods of integrating problem (1)–(3):

$$\begin{array}{ll} 1) & \Delta u_{i,k} = \Delta^2 u_{i,k} \\ 2) & \Delta u_{i,k} = \Delta^2 u_{i,k+1} \end{array} \left\{ \begin{array}{l} u_{i,0} = f(ih); \qquad u_{0,k} = u_{n,k} = 0 \\ (i = 1, 2, \dots, n-1;\ k = 0, 1, \dots, m-1). \end{array} \right.$$

The "explicit" method 1) is simple, but for stability it requires in practice a stringent restriction on the step l, namely $l \leq h^2/2$.

The "implicit" method 2) is considerably more complicated than method 1), since for each k it requires the solution of a system (though with a Jacobi matrix) of linear algebraic equations of order $(n-1)$. However, method 2) is stable (in the sense of nonaccumulation of errors) for any relation between l and h^2, and therefore in practice it is preferred to method 1).

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In this note we propose a method that is just as simple as method 1) and, at the same time, is stable under a weaker restriction on the step l. Schematically, the proposed method may be represented as follows: $**$ for even k and $**$ for odd k (or conversely) instead of $**$ and $**$ in the cases of methods 1) and 2), respectively.

The numerical method discussed in this note is written in the form of the following functional equations:

$$F_{i,2k+1} = \frac{1}{1+\omega}\left[F_{i-1,2k+1} + F_{i+1,2k} - (1-\omega)F_{i,2k}\right], \tag{4}$$

$$F_{i,0} = f_{i,0}, \qquad F_{0,k} = F_{n,k} = 0$$

for the scheme $**$ (the nodes along the x-axis are counted from left to right) and

$$F_{i,\,2k+2} = \frac{1}{1+\omega}\left[F_{i+1,\,2k+2} + F_{i-1,\,2k+1} - (1-\omega)F_{i,\,2k+1}\right], \tag{5}$$

$$\left[F_{i,0} = f_{i,0}\right], \qquad F_{0,k} = F_{n,k} = 0$$

for the scheme $**$ (the nodes along the x-axis are counted from right to left).

In the case $\omega = 1$, i.e. $h^2 = l$, equations (4) and (5) take the particularly simple form:

$$F_{i,\,2k+1} = \frac{1}{2}\left(F_{i-1,\,2k+1} + F_{i+1,\,2k}\right); \tag{6}$$

$$F_{i,\,2k+2} = \frac{1}{2}\left(F_{i-1,\,2k+1} + F_{i+1,\,2k+2}\right) \tag{7}$$

respectively.

Equations (6) and (7) are equivalent to the following single equation, which makes it possible, for any i $(1 \le i \le n-1)$, to compute $F_{i,\,2k+2}$ directly from $F_{j,\,2k}$ $(1 \le j \le n-1)$ and the boundary conditions:

$$\begin{aligned} F_{i,\,2k+2} = {} & \frac{1}{3}\left(1 - \frac{1}{2^{2(n-i)}}\right)\left(\sum_{\gamma=0}^{i-2}\frac{1}{2^{\gamma}}F_{i-\gamma,\,2k} + \frac{1}{2^{i-2}}F_{0,\,2k+1}\right) \\ & + \frac{1}{3}\sum_{\gamma=1}^{n-i-1}\frac{1}{2^{\gamma}}\left(1 - \frac{1}{2^{2(n-i-\gamma)}}\right)F_{i+\gamma,\,2k} + \frac{1}{2^{n-i}}F_{n,\,2k+2}. \end{aligned} \tag{8}$$

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The error that arises from replacing equation (1) by equation (8) (or, equivalently, by equations (6) and (7)) is determined by the quantity

$$\tau_{i,\,2k+2}^{(h)} h + O(h^2),$$

where

$$\tau_{i,\,2k+2}^{(h)} = \frac{\partial^3 u_{i,\,2k+2}/\partial x^3}{3 \cdot 2^{\,n-1}\,(2^{\,n-i} - 1)} \cdot \frac{1}{\dfrac{1}{2^{2(n-i)} - 1}} \tag{9}$$

whence it is clear that if $\partial^3 u/\partial x^3$ at the point $x = ih,\ t = (2k+2)l$ is bounded in absolute value, then

$$\tau_{i,\,2k+2}^{(h)} \to 0 \qquad \text{as } h \to 0.$$

If, for all k, one uses either only equation (4) or only equation (5), then the error will have the form $O(h)$, i.e. it will be greater than in the case of simultaneous (successive) use of equations (4) and (5). In matrix form, equations (4), (5) will be

$$AF^{(2k+1)} = BF^{(2k)}, \qquad F^{(0)} = u^{(0)}, \tag{10}$$

$$A^* F^{(2k+2)} = B^* F^{(2k+1)}, \qquad k = 0, 1, \dots, \left[\frac{m-2}{2}\right], \tag{11}$$

where A^* and B^* are the matrices transposed respectively to A and B.

Since the matrices A and A^* are bidiagonal, the method (10), (11), like method 1), is an explicit method.

In order that the numerical algorithm (10), (11) be stable, i.e. that the rounding errors on the $(2k)$-th layer not increase in passing to the $(2k+2)$-th layer, it is clearly sufficient (we assume that the rounding errors in the initial and boundary conditions are zero) that the condition

$$\left\|A^{-1}B\right\| \le 1, \tag{12}$$

which, as is immediately verified, will certainly be satisfied if $l \ll 2h^2$.

Table 1 gives the propagation of a unit error for methods 1) and (6). (For this method of investigation, see (1).)

Table 1

	$i-4$	$i-3$	$i-2$	$i-1$	i	$i-4$	$i-3$	$i-2$	$i-1$	i
k					ε					ε
$k+1$				ε	$-\varepsilon$				0.5ε	0.25ε
$k+2$			ε	-2ε	3ε			0.25ε	0.25ε	0.1875ε
$k+3$		ε	-3ε	6ε	-7ε		0.125ε	0.1875ε	0.1875ε	0.15625ε
$k+4$		-4ε	10ε	-16ε	19ε	0.0625ε	0.125ε	0.15625ε	0.15625ε	0.13672ε

It is seen from the table that, whereas in the case of the classical scheme 1) a unit error after 4 steps along the t-axis has increased 19-fold, in the case of scheme (6), under the same conditions, the error has not only failed to increase, but has even decreased by approximately 7 times.

We now give the scheme of the proof of convergence of the method (10)–(11), at least for the case in which condition (12) is satisfied. Introducing the vector error $\varepsilon^{(k)} = u^{(k)} - F^{(k)}$, from (10), (11) we have

$$\varepsilon^{(2k+2)} = C\varepsilon^{(2k)} + \delta^{(2k)}, \qquad C = A^{*-1}B^{*}A^{-1}B,$$

whence it follows that

$$\left\|\varepsilon^{(2k+2)}\right\| \leq \|C\| \left\|\varepsilon^{(2k)}\right\| + \left\|\delta^{(2k)}\right\|, \tag{13}$$

where $\left\|\delta^{2k}\right\| = O\big(\tau^{(h)}h^3\big) + O(h^4)$; $\tau^{(h)} = \max_{i,k} \tau^{(h)}_{i,\,2k+2}(\omega)$ (for $\omega = 1$, $\tau^{(h)}_{i,\,2k+2}(\omega)$ coincides with the right-hand side of (9)).

Denoting by δ the maximum norm of the vectors $\delta^{(k)}$, and taking into account that $\varepsilon^{(0)} \equiv 0$, from (13) we have

$$\left\|\varepsilon^{(2k+2)}\right\| \leq \delta \sum_{i=0}^{k} \|C\|^{i}.$$

From what has been said above the following theorem follows:

Theorem. If the solution $u(x,t)$ of problem (1)–(3) has fourth-order derivatives bounded in absolute value in the domain $0 \leq x \leq 1$, $0 \leq t \leq T$, then the approximating problem (4), (5), at least for $\omega \geq {}^{1}/2$, is stable, and its solution –the function $F_{i,k}$–converges uniformly in $0 \leq x \leq 1$, $0 \leq t \leq T$ to $u(x,t)$ with rate

$$O\big(\tau^{(h)}h\big) + O(h^2).$$

The practical advantage of the method (4), (5) in comparison with methods 1) and 2) can occur only for sufficiently small h and sufficiently large T, i.e., in those cases where stability considerations, and not convergence considerations, are the dominant factor.

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It goes without saying that the new method can be applied to more general parabolic equations (and systems), and also extends, with the corresponding modifications, to the case where, in addition to t, there are two further independent variables.

In conclusion I express my gratitude to S. L. Sobolev for valuable suggestions.

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CITED LITERATURE

1. V. S. Ryabenkii, A. F. Filippov, *On the stability of difference equations*, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.

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