OPERATOR-ANALYTIC FUNCTIONS OF ONE INDEPENDENT VARIABLE

Let

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

M. K. FAGE

OPERATOR-ANALYTIC FUNCTIONS OF ONE INDEPENDENT VARIABLE

(Presented by Academician V. I. Smirnov on 25 IX 1956)

Let

$$L=\frac{d^n}{dx^n}+p_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}+\dots+p_0(x)I, \qquad a<x<b, \tag{1}$$

be an ordinary linear differential operator with continuous complex (in particular, real) coefficients, defined on an interval (a,b) of the real number line.

§ 1. Using the terminology already employed (1), we shall call a complex-valued function $f(x)$, defined on (a,b), infinitely L-differentiable on (a,b) if, for every $q=0,1,2,\dots$, the function $L^q f(x)$ is n times continuously differentiable on (a,b). We shall call such a function L-analytic on (a,b) if for every closed interval $[\alpha,\beta]\subset(a,b)$ there exists a constant $C>0$ (depending on $\alpha,\beta,f(x),L$) such that, for all $r=0,1,\dots,n-1$ and $q=0,1,2,\dots$, the inequalities hold ($D=d/dx$):

$$\left|D^{rL^q}f(x)\right|\le C^{qn+r}(qn+r)!, \qquad \alpha\le x\le\beta. \tag{2}$$

To study the structure of L-analytic functions, we associate with each point $x_0\in(a,b)$ an L-basis at this point, i.e. a sequence of functions $f_0(x,x_0), f_1(x,x_0),\dots$, defined on the whole interval (a,b) as follows: the first n of them are solutions of the homogeneous equation $L[f_m(x,x_0)]=0$ $(m=0,1,\dots,n-1)$ with initial values at the point x_0 forming the identity matrix; the subsequent ones are defined recursively, i.e. if $f_m(x,x_0)$ has already been defined, then $f_{m+n}(x,x_0)$ is the solution of the nonhomogeneous equation $L[f_{m+n}(x,x_0)]=f_m(x,x_0)$ with zero initial conditions at the point x_0.

With the aid of Green' s functions one can obtain the following estimates of the functions of the L-basis, fundamental for the theory of L-analytic functions.

Theorem 1. For every closed interval $[\alpha,\beta]\subset(a,b)$ there exists a constant $C_1>0$ such that

$$|f_m(x,x_0)|\le C_1|x-x_0|^m(m!)^{-1}$$

for all $m = 0, 1, 2, \ldots$ and $\alpha \le x \le \beta$.

Somewhat generalizing the scheme for constructing the theory of ordinary Taylor series (in the real domain), we establish the validity of the following theorems.

Theorem 2. In order that a function $f(x)$ be L-analytic on (a, b), it is necessary and sufficient that it be L-holomorphic on (a, b), i.e. that in a neighborhood of every point $x_0 \in (a, b)$ it be expandable in an L-series

$$f(x) = \sum_{m=0}^{\infty} a_m f_m(x, x_0), \tag{3}$$

whose coefficients satisfy the inequalities

$$|a_m| \le C_2^m m! \tag{4}$$

Theorem 3. The coefficients a_m of the series (3) are determined by the formulas $a_{qn+r} = D^{rL^q} f(x)|_{x=x_0}$, i.e., every L-series is an "L-Taylor series."

Theorem 4. An L-series admits termwise L-differentiation any number of times.

Theorem 5. The interval of convergence of an L-series in which it admits termwise L-differentiation contains the interval

$$(x_0 - R, x_0 + R) \cap (a, b), \quad \text{where } (eR)^{-1} = \overline{\lim_{m\to\infty}} \left(m^{-1} \sqrt[m]{|a_m|} \right).$$

In what form the first Abel theorem is valid is unknown. There is also a uniqueness theorem in all three usual variants:

Theorem 6. A function $f(x)$, L-analytic on (a, b), will be identically equal to zero on (a, b) if it satisfies at least one of the following sufficient uniqueness conditions: a) it is equal to zero on some interval $(\alpha, \beta) \subset (a, b)$; b) all its L-Taylor coefficients are equal to zero at at least one point $x_0 \in (a, b)$; c) it vanishes on a sequence of points tending to an interior point of the interval (a, b).

§ 2. The following theorem is also true.

Theorem 7. If L has analytic coefficients, then L-analyticity of a function on (a, b) coincides with ordinary analyticity.

Remark. If the coefficients of the operator L are analytic functions of the complex variable w in some domain G, then a function $f(w)$, analytic in G, will also be L-analytic in G in the sense of satisfying inequalities of type (2) in every closed domain $\overline{G}_1 \subset G$. Theorems 2–6 carry over to such functions; moreover, the L-basis for each point $w_0 \in G$ is defined analogously to the preceding one.

§ 3. Let $x_0 \in (a, b)$; denote by A_{L,x_0} the set of functions each of which is defined and L-analytic in some (its own) neighborhood of the point x_0. The topology in A_{L,x_0} is **regular convergence**. (A sequence of functions $f_m(x) \in A_{L,x_0}$ $(m = 0, 1, \ldots)$ will be called regularly convergent if the following conditions are fulfilled: a) there is a common neighborhood U of the point x_0 in which all $f_m(x)$ are defined; b) each D^{rL^q}-derivative $D_m^{rL^{qf}}(x)$ tends in U to some limit; c) there exists a constant $C > 0$ with estimates of the form (2) for all $x \in U$, $r = 0, 1, \ldots, n-1$, and $q, m = 0, 1, 2, \ldots$.)

Let

$$M = \frac{d^n}{dw^n} + \cdots + q_0(w)I$$

be an operator whose coefficients are analytic functions of the complex variable w, varying in some domain G. Take any point $w_0 \in G$ and denote by A_{w_0} the set of functions each of which is defined and analytic (i.e., on the basis of the remark in § 2, M-analytic) in some (its own) neighborhood of w_0. The topology in A_{w_0} is defined as uniform convergence in some neighborhood of w_0 (its own for each convergent sequence of functions); by virtue of the Cauchy integral formulas it is equivalent to regular convergence with respect to any operator M with analytic coefficients.

Now, following A. Ya. Povzner (^{2}), who applied this method to differential operators of the second order, we consider the equation

$$M[F(w, x)] = L[F(w, x)] \tag{5}$$

and, taking some function $h(w) \in A_{w_0}$, we solve, for equation (5), for each $k = 0, 1, \ldots, n-1$, the following k-th Cauchy problem.

Find a solution $F(w, x) = F_k(w, x)$ of equation (5) satisfying, for $x = x_0$, the following initial conditions:

$$\left.\frac{\partial^s F(w, x)}{\partial x^s}\right|_{x=x_0} = \begin{cases} 0, & s \neq k, \\ h(w), & s = k. \end{cases} \qquad (0 \leq s \leq n-1) \tag{6}$$

For the method of solving this problem see (^{3,4}).

We now define the transformation $T = T_{M,w_0;\,L,x^0}$ by the formula

$$f(x) = T[h(w)] = \sum_{k=0}^{n-1} \left.\frac{\partial^k F_k(w, x)}{\partial w^k}\right|_{w=w_0}. \tag{7}$$

One can prove the following theorem.

Theorem 8. The transformation T carries the function $h_m(w, w_0)$ of the M-basis into the function $f_m(x, x_0)$ of the L-basis ($m = 0, 1, ...$), and every function

$$h(w) = \sum_{m=0}^{\infty} a_m h_m(w, w_0) \in A_{w_0},$$

with M-Taylor coefficients a_m (satisfying inequalities (3)), into the function

$$f(x) = \sum_{m=0}^{\infty} a_m f_m(x, x_0) \in A_{L,x_0}$$

with the same L-Taylor coefficients; T is continuous.

Suppose, in addition, that the operator L has an adjoint operator, i.e. the coefficient $p_k(x)$ is continuously differentiable k times ($k = 0, 1, \dots, n-1$). Then the generalized Riemann method (^{3,4}) may be applied to the solution of the k-th Cauchy problem (5), (6), and formula (7) may be given the form of an integral transformation, which we do not write down for lack of space. For $n = 2$ the corresponding formula becomes the formula of the "transformation operator," found, investigated, and applied in various ways in many works.

§ 4. Let L and L_1 be two operators of the form (1), with coefficients continuous on the intervals (a, b) and, respectively, (a_1, b_1) (or one of them, or both, has coefficients analytic in some complex domain G). Taking a point $x_0 \in (a, b)$, $x_1 \in (a_1, b_1)$ (or $w_0 \in G$, etc.), construct the L-basis $\{f_m(x, x_0)\}$ and the L_1-basis $\{g_m(x, x_1)\}$, and define the transformation $T = T_{L_1,x_1;\,L,x_0}$ by the formulas $T[g_m] = f_m$ ($m = 0, 1, 2, ...$), extending it linearly to the whole set A_{L_1,x_1}. We obtain a one-to-one and mutually continuous mapping of the linear topological spaces A_{L_1,x_1} onto A_{L,x_0}; moreover, by the definition of the L- and L_1-bases, if $f = T[g]$, then $f(x_0) = g(x_1), \dots, f^{(n-1)}(x_0) = g^{(n-1)}(x_1)$ (the property of "preservation of initial data") and $L[f] = T[L_1[g]]$ (the property of "transformation of operators"); hence $L = TL_1T^{-1}$, i.e. Theorem 9 holds.

Theorem 9. Any two ordinary linear differential operators of the same order with continuous coefficients are locally equivalent.

The word "locally" indicates the circumstance that each set A_{L,x_0} contains functions L-analytic both in large and in (arbitrarily) small neighborhoods of the point x_0.

Thus, we obtain a direct construction of the transformation $T = T_{L_1,x_1;\,L,x_0}$, without recourse to the theory of partial differential equations—an apparatus foreign to this simple fact of equivalence of all ordinary linear differential operators of the same order. Partial differential equations were brought in (in § 3) in order to give this equivalence an integral form, since, as Theorem 8 shows, the transformation $T_{M,w_0;\,L,x_0}$ in § 3 is precisely the transformation $T_{L_1,x_1;\,L,x_0}$ in the special case when $L_1 = M$, $x_1 = w_0$.

On the contrary, the theory of L-analytic functions makes it possible to characterize more fully, for example, the solution of the k-Cauchy problem (5), (6); namely, as a function of x, it is an L-analytic function.

Let us note that A_{L,x_0} can be made into a commutative topological ring by defining the convolution (depending on x_0) of two functions from A_{L,x_0} in terms of their L-Taylor coefficients by the formula

$$\frac{c_m}{m!} = \sum_{k+l=m} \frac{a_k}{k!} \frac{b_l}{l!} \quad (m = 0, 1, 2, ...). \tag{8}$$

Then the transformation T preserves convolution.

Finally, one can obtain a canonical form of all operators L by considering simply the ring A of elements—complex sequences $\tilde{a} = (a_0, a_1, ...)$ with multiplication (8) and with the topology of coordinate convergence under the simultaneous fulfillment of inequalities (3) (with a constant C_2 depending on each convergent sequence of elements). Defining the operator Λ in the ring A as the left shift, $\Lambda\tilde{a} = (a_1, a_2, ...)$, we obtain Theorem 10.

Theorem 10. *All rings A_{L,x_0} are isomorphic to A. All ordinary linear differential operators of order n with continuous coefficients are (locally) equivalent to Λ^n.*

Chernivtsi
State University

Received
18 VIII 1956

REFERENCES

[1] A. G. Kostyuchenko, DAN, 98, No. 1 (1954).
[2] A. Ya. Povzner, Matem. sborn., 23 (65), No. 1 (1948).
[3] M. K. Fage, Proceedings of the 3rd All-Union Mathematical Congress, 1, 71-72 (1956).
[4] M. K. Fage, DAN, 108, No. 5 (1956).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.