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Abstract

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DIFFERENCE SCHEMES FOR MULTIDIMENSIONAL PROBLEMS

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In this note we wish to indicate a new method for constructing and studying the stability of difference schemes for solving Cauchy problems for multidimensional systems of hyperbolic equations of the form

$$\frac{\partial u_i}{\partial t} = \sum_{j=1,\ k=1}^{j=m,\ k=n} a_{ij}^k \frac{\partial u_j}{\partial x_k} + \sum_{j=1}^{m} b_{ij} u_j. \tag{T}$$

We shall regard the coefficients $a_{ij}^k,\ b_{ij}$ as constant, although the principles proposed here can also be applied to equations with variable coefficients.

In addition to the system (T), let us consider n auxiliary systems

$$\frac{\partial u_i^{(k)}}{\partial t} = \sum_{j=1}^{m} a_{ij}^k \frac{\partial u_j^{(k)}}{\partial x_k} + \sum_{j=1}^{m} b_{ij}^k u_j^{(k)}, \qquad k = 1,\ 2, \dots,\ n \tag{T_k}$$

$$\left(b_{ij}^k \text{ are arbitrary, satisfying the condition } \sum_{k=1}^{n} b_{ij}^k = b_{ij} \right).$$

In each of these systems the unknown functions $u_i^{(k)}$ depend on the time t and on only one spatial variable x_k. Our aim will be to show that if we are able to construct difference schemes for the "one-dimensional" systems (T_k), then we shall be able to construct a certain scheme also for the "n-dimensional" system (T). As is known ([1], Theorem 1, p. 28), two requirements must be imposed on difference schemes for solving well-posed problems for differential equations. They must be: a) locally exact and b) stable. In this case, as the steps of the difference mesh are refined, the solutions of the difference equations will tend to the solutions of the differential equations. To justify the reasonableness of the schemes we propose, we shall verify that they satisfy the requirements of the theorem cited.

Let us consider the hyperbolic system of differential equations

$$\frac{\partial u_i}{\partial t} = \sum_j a_{ij}\frac{\partial u_j}{\partial x} + \sum_j b_{ij}u_j. \tag{D}$$

We impose the requirement of hyperbolicity in order to ensure the well-posedness of the Cauchy problem.

Take a rectangular mesh in the x, t plane with step τ in time t and step h in the spatial variable x. The value of the function u_i at the mesh point $t = p\tau,\ x = qh$ will be denoted by

$$u_i(x,t) = u_i(qh, p\tau) = u_i^{p,q}.$$

Suppose that we have constructed a certain difference scheme of the form

$$\tilde{u}_i^{p,q} = \sum_{l,j} \widetilde{C}_{ij}^{l} u_j^{p-1,q+l}. \tag{$\widetilde{R}$}$$

In what follows we shall write it as

$$\tilde{u}^p = \tilde{C}\tilde{u}^{p-1},$$

understanding by $\tilde{C}$ the matrix that transforms the infinite-dimensional vector $\tilde{u}^{p-1} = \{\tilde{u}_i^{p-1,q}\}$ into the vector $\tilde{u}^p$.

Suppose that $(\widetilde{R})$ approximates the system (D) and that the scheme is stable in the sense that, for sufficiently small τ,

$$\|\tilde{u}^p\| \le (1 + k\tau)\|\tilde{u}^{p-1}\|,$$

or, what is the same,

$$\|\tilde{C}\| \le (1 + k\tau),$$

where by the norm of the vector $\tilde{u}^m$ we mean

$$\|\tilde{u}^m\| = \left(\sum_{i,q} g_i |\tilde{u}_i^{m,q}|^2\right)^{1/2}.$$

The vector whose components are the positive numbers g_i will be denoted by g. A scheme possessing the indicated properties will be called **g-stable**. We shall show that if, for each of the systems (T_k), we can construct a g-stable scheme (the vector g being common for all k), then we can construct a stable scheme also for the system (T). Let the difference scheme (R_k) for (T_k) be defined by the matrix C_k.

Consider now u_i, depending on the $n+1$ arguments $t, x_1, x_2, \dots, x_n$, and denote $u_i(p\tau, q_1 h_1, q_2 h_2, \dots, q_{nh} n)$ by $u_i^{p, q_1 \cdots q_n}$. We shall say that $u_i^{p, q_1 \cdots q_n}$ satisfies (R_k) if

$$u_i^{p, q_1 \cdots q_n} = \sum_{j,l} C_{i, j(k)}^{\,l - q_k} u_j^{p-1, q_1 \cdots q_{k-1}, \, l, \, q_{k+1} \cdots q_n}.$$

For short, we shall write this in the form

$$u^p = C_k(\tau, h_k) u^{p-1}.$$

We assert that the difference scheme

$$u^p = F(\tau, h_1 \dots h_n) u^{p-1}, \quad \text{where } F = C_n(\tau, h_n) C_{n-1}(\tau, h_{n-1}) \dots C_1(\tau, h_1) \quad \text{(R)}$$

will be stable. It is not difficult to verify that it will approximate the system (T).

To verify stability, define

$$\|u^p\| = \|\{u^{p, q_1 \cdots q_n}\}\| = \left(\sum_{i, q_1 \cdots q_n} g_i |u_i^{p, q_1 \cdots q_n}|^2 \right)^{1/2}.$$

It is obvious that each of the matrices C_k increases this norm by no more than a factor $1 + k\tau$, and consequently the same will be true for their product (only with another constant k).

With the aid of "one-dimensional" schemes one can construct "multidimensional" schemes of a form different from that of scheme (R). For example, if the matrices C_k can be represented in the form

$$C_k = E + \frac{\mu_k \tau}{h_k} H_k *,$$

where E is the identity matrix; μ_k is some numerical parameter such that, for $\mu_k \tau / h_k \ll 1$, the scheme is g-stable; and H_k is a matrix formally independent of τ and h_k (it is clear that, for sufficiently smooth functions u, H_{ku} must tend to zero as $h_k \to 0$), then one can construct

[* Here we assume, for simplicity, that in the systems (T_k) all coefficients b_{ij}^k are equal to zero.

the following difference scheme approximating (T):

$$u^p = G u^{p-1}, \qquad \text{where } G = E + \sum_k \frac{\mu_k \tau}{h_k} H_k.$$

This scheme will certainly be stable when

$$\sum_k \frac{\mu_k \tau}{h_k} \ll 1,$$

as follows from the inequality

$$\|G\| \leq \left| 1 - \sum_k \frac{\mu_k \tau}{h_k} \right| \|E\| + \sum_k \frac{\mu_k \tau}{h_k} \|E + H_k\|.$$

Let us note that scheme (R) in this case would be stable under

$$\max_k \frac{\mu_k \tau}{h_k} \ll 1,$$

i.e., in a somewhat larger domain. Moreover, scheme (R), computation by which reduces to the successive execution of one-dimensional computations, may prove more convenient for computing machines.

As an example, let us consider the application of the facts noted above to the construction of a difference scheme for the simplest system of hyperbolic equations

$$\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = 0, \qquad \frac{\partial v}{\partial t} + \frac{\partial p}{\partial y} = 0, \qquad \frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

As the "one-dimensional" scheme with respect to the variable x, one may take here, for example,

$$u(t+\tau, x) = u(t, x) -$$

$$-\frac{\tau}{2h}\left[p(t, x+h) - p(t, x-h)\right] + \frac{\tau}{2h}\left[u(t, x+h) - 2u(t, x) + u(t, x-h)\right],$$

$$v(t+\tau, x) = v(t, x),$$

$$p(t+\tau, x) = p(t, x) - \frac{\tau}{2h}\left[u(t, x+h) - u(t, x-h)\right] +$$

$$+\frac{\tau}{2h}\left[p(t,x+h)-2p(t,x)+p(t,x-h)\right].$$

This scheme does not increase the norm defined as the sum, over all grid points, of the quantities $u^2+v^2+p^2$. The one-dimensional scheme with respect to the variable y is constructed analogously.

Let us note that the representation of difference schemes for complicated systems of equations in the form of a “product”of schemes composed for individual groups of terms entering these systems may prove convenient in a number of other cases as well. Thus, for example, a difference scheme for the system

$$\frac{\partial u}{\partial t}-A\frac{\partial v}{\partial x}+B\frac{\partial T}{\partial x}=0,\qquad \frac{\partial v}{\partial t}-\frac{\partial u}{\partial x}=0,\qquad \frac{\partial T}{\partial t}+C\frac{\partial u}{\partial x}=\frac{\partial^2 T}{\partial x^2},$$

describing the propagation of sound in a heat-conducting medium, can be constructed as the “product” of difference schemes for two simpler systems (1) and (2):

$$\frac{\partial u}{\partial t}-A\frac{\partial v}{\partial x}+B\frac{\partial T}{\partial x}=0,\qquad \frac{\partial v}{\partial t}-\frac{\partial u}{\partial x}=0,\qquad \frac{\partial T}{\partial t}+C\frac{\partial u}{\partial x}=0; \tag{1}$$

$$\frac{\partial u}{\partial t}=0,\qquad \frac{\partial v}{\partial t}=0,\qquad \frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}. \tag{2}$$

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CITED LITERATURE

1. V. S. Ryabenkii, A. F. Filippov, *On the stability of difference equations*, Moscow, 1956.

Note: Figure translations are in progress. See original paper for figures.

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