ON THE PROPERTIES OF SOME \(\delta s\)-OPERATIONS

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Abstract

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I. D. STUPINA

ON THE PROPERTIES OF SOME δs-OPERATIONS

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In descriptive set theory a number of results of the following character have been obtained: the projection of a plane set is considered, and assertions are made about the descriptive nature of the set of points of the projection whose inverse images possess a certain special property. For example: if a plane A-set is projected, then the set of points whose inverse images contain at least two points is an A-set (N. N. Luzin); the set of points whose inverse images contain an uncountable set of points is likewise an A-set (W. Sierpiński).

A number of problems of the same character were solved by P. S. Novikov ([3]), V. Ya. Arsenin ([4,5]), C. Braun ([6]), K. Kunugui ([7]). Analogous theorems in the theory of operations on sets were established by A. A. Lyapunov ([8,9]) and Z. I. Kozlova ([10,11]).

In the present note some results in this direction are given*.

Let N be a rigid base** of some δs-operation. We shall call a point x a point of N-first type for the sequence of sets $\{E_n\}$ if there exists an uncountable number of chains of the base N into whose kernels the point x enters. By Φ_{N***} we denote the δs-operation which selects the points of N-first type. If each chain of an N-rigid base of a δs-operation is ordered in increasing order of the elements of the chain, then the resulting collection of chains will be called the rigid reduced base and denoted by $\check{N}$.

A point x is determined by the chain $\eta = \{n_i\}$ with respect to the sequence of sets $\{E_n\}$, if

$$x \in \prod_{n_i \in \eta} E_{n_i}.$$

Let $M_x = \{\eta\}$ denote, for the given sequence of sets $\{E_n\}$ and base N, the set of all such chains $\eta \in \check{N}$ that x is determined by the chain η with respect to the sequence of sets $\{E_n\}$. We shall call a point x, determined by the δs-operation Φ_N, a point of N-second type if the set M_x has non-compact closure. By $\Phi_{\check{N}}$ we denote the δs-operation selecting the points of N-second type.

We shall call a point x, determined by the δs-operation Φ_N, a point of N-third type of order α, if the set M_x is not a scattered set of index $\leqslant \alpha$. By $\Phi_{\check{N}\alpha}$ we denote the δs-operation selecting the points of N-third type of order α.

We shall consider linear sets E which are sums of a scattered family of sets whose closure is compact.

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* We use the notation introduced in the works $(^{8-11})$.

** In what follows we shall consider δs-operations Φ_N with rigid bases N belonging to the Baire space J.

Let $E = E^{(0)}$. If α is a transfinite number of the first kind, then $E^{(\alpha)}$ denotes the set obtained from the set $E^{(\alpha-1)}$ by removing all its isolated portions whose closures are compact; if α is a transfinite number of the second kind, then, by definition,

$$E^{(\alpha)} = \prod_{\alpha' < \alpha} E^{(\alpha')}.$$

The least number β such that $E^{(\beta)} = 0$ is called the **index** of the set E.

We shall call a point x, determined by the δs-operation Φ_N, a point of N**-fourth type of order** α, if the set M_x is not the sum of a dispersed family of sets whose closures are compact, of index $\leqslant \alpha$. By $\Phi_{\check{N}_*^{(\alpha)}}$ we shall denote the δs-operation selecting points of N-fourth type of order α.

Z. I. Kozlova showed $(^{11})$ that if N is a rigid base of an A-operation, then the operations $\Phi_{N^{***}}, \Phi_{\check{N}_*}, \Phi_{\check{N}_\alpha}, \Phi_{\check{N}_*^{(\alpha)}}$ are no more powerful than A-operations.

Theorem 1. *The operations $\Phi_{N_c'^{***}}, \Phi_{N_c'^{***n}}$ are no more powerful than the Γ-operation, where N_c' is a rigid base of the Γ-operation.*

Theorem 2. *The operations $\Phi_{N_c''^{***}}, \Phi_{N_c''^{***n}}$ are no more powerful than the CA_2-operation, where N_c'' is a rigid base of the CA_2-operation.*

Theorem 3. *If the class of sets Ξ and the rigid base N are in a completely regular relation, then the relations*

$$\Phi_{\check{N}_*}(\Xi) \subset \Phi_N(\Xi), \qquad \Phi_{\check{N}_*^n}(\Xi) \subset \Phi_N(\Xi)$$

hold.

Theorem 4. *If the class of sets Ξ is invariant with respect to the δs-operations Φ_N, where the base N is an A_2-set, then*

$$\Phi_{\check{N}''_\alpha}(\Xi) \subset \Xi.$$

If, however, the class of sets Ξ^* is such that $\Phi_{N''}(\Xi^*)$ belongs to the class of A_2-sets Θ, then

$$\Phi_{\check{N}''_{\alpha}}(\Xi^*) \subset \Theta,$$

where N'' is a rigid base of the A_2-operation.

Theorem 5. *If the class of sets Ξ is invariant with respect to the δs-operations Φ_N, where the base N is an A_2-set, then*

$$\Phi_{\check{N}_*''(\alpha)}(\Xi) \subset \Xi.$$

If, however, the class of sets Ξ^* is such that $\Phi_{N''}(\Xi^*)$ belongs to the class of A_2-sets Θ, then

$$\Phi_{\check{N}_*''(\alpha)}(\Xi^*) \subset \Theta.$$

From the general theorem on covering sets of Z. I. Kozlova (([11]), theorem 1) and the preceding results, the following propositions follow:

Theorem 6. *For every sequence of CA_2-sets $\{E_n\}$ such that no point of the set $\Phi_N\{E_n\}$ is a point of N-first type, there exists a sequence of B_2-sets $\{H_n\}$ such that $H_n \supset E_n$ and no point of the set $\Phi_N\{H_n\}$ is a point of N-first type, where N is a rigid base of a Γ-operation, of a CA_2-operation.*

Theorem 7. For every sequence of CA_2-sets $\{E_n\}$ such that no point of the set $\Phi_N\{E_n\}$ is a point of the second N-type, there exists a sequence of B_2-sets $\{H_n\}$ such that $H_n \supset E_n$ and no point of the set $\Phi_N\{H_n\}$ is a point of the second N-type, where N is a rigid base of the Γ-operation, a CA_2-operation.

By virtue of a theorem of P. S. Novikov [13] stating that, in the system Σ of axioms of K. Gödel' s set theory [12], the separation theorems for projective sets CA_n, for sufficiently large n, are not contradictory, one may, in the sense of consistency, formulate analogous propositions on covering CA_n-sets.

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Note: Figure translations are in progress. See original paper for figures.