THEORY OF ELASTICITY

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Abstract

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ON A PROBLEM IN THE THEORY OF ELASTICITY WITH MIXED HOMOGENEOUS CONDITIONS

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§ 1. Suppose that an elastic, isotropic, and homogeneous medium fills a finite simply connected domain S, situated in the plane of the complex variable $z = x + iy$, bounded by a sufficiently smooth closed contour L. As the origin of coordinates we take a point belonging to the domain S. The contour is traversed, as usual, counterclockwise. Assume that on the boundary L the normal component of the displacement vector v_n and the tangential component of the stress vector T are prescribed, and it is required to find the components of the stress tensor and of the displacement vector which arise in the medium. We have already considered this problem earlier for the general case of a multiply connected domain ([1]); for it a uniquely solvable system of Fredholm integral equations was obtained. Unfortunately, the kernels of this system were found in the form of certain quadratures and, generally speaking, cannot be expressed through elementary or known tabulated functions; this circumstance, naturally, complicates the practical use of the named Fredholm system. In the present article, restricting ourselves for simplicity to the case of a simply connected domain S, we reduce the problem under consideration to a new, much more convenient, system of Fredholm equations; its kernels, as we shall see below, are expressed directly through elementary functions. With the aid of the resources of modern computational technique it is comparatively easy to carry out a numerical interpretation of such a system of integral equations.

The prescribed boundary values of the quantities v_n and T, in turn, are expressed in the known way through two functions $\varphi(z)$ and $\psi(z)$ of the complex variable z, regular in the domain S ([2]); the solution of the problem reduces to determining them from two real boundary conditions. It seems expedient to change somewhat the form of the second limiting equality, which relates the value T to the indicated functions. Namely, the first limiting equality, expressing v_n through the same functions $\varphi(z)$ and $\psi(z)$, we differentiate with respect to the arc s, measured from some fixed origin on L, and, multiplying by twice the modulus of shear μ, add term by term to the second equality. Then we put in them $\varphi(z) = u(x, y) + iv(x, y)$, $\psi(z) = p(x, y) + iq(x, y)$, where u, v and

p, q are, respectively, harmonically conjugate functions. After this the limiting equalities will take the form:

$$\delta_1(s)\frac{\partial u}{\partial \xi} + \delta_2(s)\frac{\partial v}{\partial \xi} + \chi(\dot{\eta}u - \dot{\xi}v) + \dot{\eta}p + \dot{\xi}q = f_1(s), \tag{1}$$

$$\gamma_1(s)\frac{\partial u}{\partial \xi} + \gamma_2(s)\frac{\partial v}{\partial \xi} + \chi(\ddot{\eta}u - \ddot{\xi}v) + \ddot{\eta}p + \ddot{\xi}q = f_2(s). \tag{2}$$

Here the meaning of the notation introduced is as follows: $t = \xi + i\eta$ is the affix of a point lying on the curve L; χ is an elastic constant; one or two dots placed above the coordinates denote single or double differentiation with respect to the arc s; moreover, the coefficients $\delta_1, \dots, \gamma_2$ and the free terms $f_1(s)$ and $f_2(s)$ are known functions of the arc, equal to

$$\delta_1(s) = \dot{\xi}\eta - \eta\dot{\xi}, \qquad \delta_2(s) = \xi\dot{\xi} + \dot{\eta}\eta, \qquad \gamma_1(s) = \dot{\xi}\ddot{\eta} - \eta\ddot{\xi}, \tag{3}$$

$$\gamma_2(s) = \xi\ddot{\xi} + \eta\ddot{\eta} - (\varkappa - 1), \qquad f_1(s) = 2\mu v_n, \qquad f_2(s) = 2\mu\frac{\partial v_n}{\partial s} + T.$$

We shall seek the functions $\varphi(z)$ and $\psi(z)$ in the form

$$\varphi(z) = \sum_{j=1}^{2}\int_L \nu_j(s)G_j(t,z)\,dt, \qquad \psi(z) = \sum_{j=1}^{2}\int_L \nu_j(s)H_j(t,z)\,dt, \tag{4}$$

where $\nu_j(s)$ $(j = 1, 2)$ are real densities to be determined, and

$$G_1(t,z) = -\dot{t}\ddot{t}G_2(t,z), \qquad G_2(t,z) = \frac{1}{\pi(\varkappa-1)}\left\{-1 + \ln\left(1 - \frac{z}{t}\right)\right\},$$

$$H_1(t,z) = \frac{1}{\pi(\varkappa-1)}\left\{\left[\bar{t}\dot{t}\ddot{t} + (\varkappa-1)\dot{t}\right]\left(\frac{1}{t-z} - \frac{1}{t}\right) + \varkappa\,\dot{t}\ddot{t}\ln\left(1 - \frac{z}{t}\right)\right\}, \tag{5}$$

$$H_2(t,z) = \frac{1}{\pi(\varkappa-1)}\left\{\bar{t}\left(\frac{1}{t-z} - \frac{1}{t}\right) - \varkappa\dot{t}^2\ln\left(1 - \frac{z}{t}\right)\right\},$$

where the branch of the logarithm occurring here vanishes at the origin of coordinates. Note that, by construction, $\psi(0) = 0$. It will become clear from what follows that any two functions regular in some (finite) simply connected domain are in fact representable (one of them up to an additive constant) in the form of the integrals (4), distributed over the contour bounding the domain.

In the functions (4), and in the derivative of the first of them, we pass to the limit as z tends to the point $t_0 = \xi_0 + i\eta_0$ of the contour L; the limiting values of these functions found according to the Sokhotski–Plemelj formulas are substituted into the boundary conditions (1) and (2). Then, after somewhat lengthy but sufficiently obvious transformations, we obtain, for determining the unknown densities, a system of Fredholm integral equations

$$\nu_j(s_0) + \sum_{n=1}^{2} \int_L \nu_n(s) K_{nj}(s, s_0)\, ds = f_j(s_0) \qquad (j = 1, 2). \tag{6}$$

In it the kernels, as is clear from formula (7) given below, are continuous functions of the arguments s and s_0; for clarity we split each of them into a sum of three terms

$$K_{ej}(s, s_0) = P_{ej}(s, s_0) \frac{\partial \ln r_0}{\partial n} + Q_{ej}(s, s_0) \frac{\partial \ln r_0}{\partial s} + R_{ej}(s, s_0),$$

$$r_0 = \sqrt{(\xi - \xi_0)^2 + (\eta - \eta_0)^2},$$

where n is the normal to L directed outward from S, and the following notation has been introduced ($Q_{nj}(s, s_0) = 0$ for $s = s_0$; the lower zero subscript is assigned to functions depending on the arc s_0):

$$P_{11} = \frac{1}{\pi}(\xi\dot{\xi}_0 + \dot{\eta}\eta_0) + (\dot{\xi}\ddot{\eta} - \eta\ddot{\xi}) Q_{21}, \qquad R_{11} = \frac{1}{\pi(\varkappa - 1)}(\varkappa R_{11}^{(1)} + R_{11}^{(2)}),$$

$$Q_{11} = \frac{1}{\pi}(\dot{\xi}\eta_0 - \xi_0\dot{\eta}) - (\dot{\xi}\ddot{\eta} - \eta\ddot{\xi}) P_{21}, \qquad R_{12} = \frac{1}{\pi(\varkappa - 1)}(\varkappa R_{12}^{(1)} + R_{12}^{(2)}),$$

$$P_{12} = \frac{1}{\pi}(\dot{\xi}\xi_0 + \dot{\eta}\eta_0) + (\dot{\xi}\ddot{\eta} - \eta\ddot{\xi}) Q_{22}, \qquad R_{21} = \frac{1}{\pi(\varkappa - 1)}(\varkappa R_{21}^{(1)} + R_{21}^{(2)}),$$

$$Q_{12} = \frac{1}{\pi}[\xi(\eta_0 - \eta) - \eta(\xi_0 - \xi)] - (\dot{\xi}\ddot{\eta} - \eta\ddot{\xi}) \left(P_{22} - \frac{1}{\pi}\right),$$

$$R_{22} = \frac{1}{\pi(\varkappa - 1)}(\varkappa R_{22}^{(1)} + R_{22}^{(2)}),$$

$$Q_{21} = \frac{1}{\pi(\varkappa - 1)} \left[\dot{\eta}_0(\xi - \xi_0) - \dot{\xi}_0(\eta - \eta_0)\right],$$

$$P_{21} = \frac{1}{\pi(\varkappa - 1)} \left[\dot\xi_0(\xi - \xi_0) + \dot\eta_0(\eta - \eta_0)\right],$$

$$
\begin{gathered}
P_{22} = \frac{1}{\pi}\left\{1 + \frac{1}{\varkappa - 1}\left[\ddot\xi_0(\xi - \xi_0) + \ddot\eta_0(\eta - \eta_0)\right]\right\}, \\
Q_{22} = \frac{1}{\pi(\varkappa - 1)}\left[\ddot\eta_0(\xi - \xi_0) - \ddot\xi_0(\eta - \eta_0)\right], \\
R_{11}^{(1)} = \left\{-(\dot\xi_0\ddot\eta - \dot\eta_0\ddot\xi) + 2(\dot\xi_0\ddot\xi + \dot\eta_0\ddot\eta)\left(\operatorname{arctg}\frac{\eta - \eta_0}{\xi - \xi_0} - \operatorname{arctg}\frac{\eta}{\xi}\right)\right\}, \\
R_{12}^{(1)} = (\ddot\xi\ddot\eta_0 - \ddot\xi_0\ddot\eta) + 2(\ddot\xi_0\ddot\xi + \ddot\eta_0\ddot\eta)\left(\operatorname{arctg}\frac{\eta - \eta_0}{\xi - \xi_0} - \operatorname{arctg}\frac{\eta}{\xi}\right), \\
R_{21}^{(1)} = \left\{(\dot\xi_0\dot\eta - \dot\xi\dot\eta_0) - 2(\dot\xi_0\dot\xi + \dot\eta_0\dot\eta)\left(\operatorname{arctg}\frac{\eta - \eta_0}{\xi - \xi_0} - \operatorname{arctg}\frac{\eta}{\xi}\right)\right\}, \\
R_{22}^{(1)} = -\left\{(\dot\xi\ddot\eta_0 - \dot\eta\ddot\xi_0) + 2(\dot\xi\ddot\xi_0 + \dot\eta\ddot\eta_0)\left(\operatorname{arctg}\frac{\eta - \eta_0}{\xi - \xi_0} - \operatorname{arctg}\frac{\eta}{\xi}\right)\right\}, \\
R_{11}^{(2)} = \frac{1}{\rho^2}\left\{(\varkappa - 1)(\dot\xi_0\eta - \dot\eta_0\xi) + (\dot\xi\eta - \dot\eta\xi)[\dot\xi_0 a(s) - \dot\eta_0 b(s)]\right\}, \\
R_{12}^{(2)} = \frac{1}{\rho^2}\left\{(\varkappa - 1)(\ddot\xi_0\eta - \ddot\eta_0\xi) + (\dot\xi\eta - \dot\eta\xi)[\ddot\xi_0 a(s) - \ddot\eta_0 b(s)]\right\}, \\
R_{21}^{(2)} = -\frac{1}{\rho^2}[\dot\xi_0 b(s) + \dot\eta_0 a(s)], \qquad R_{22}^{(2)} = -\frac{1}{\rho^2}[\ddot\xi_0 b(s) + \ddot\eta_0 a(s)], \\
a(s) = (\xi^2 - \eta^2)\dot\xi + 2\xi\eta\dot\eta, \qquad b(s) = (\xi^2 - \eta^2)\dot\eta - 2\xi\eta\dot\xi, \qquad \rho = \sqrt{\xi^2 + \eta^2}.
\end{gathered}
\tag{7}
$$

It is not hard to notice that system (6) remains Fredholm if, in the second of formulas (4), the terms containing the logarithmic factor are omitted from under the integral sign. The point, however, is that their presence makes it possible to study the properties of system (6) very simply. Conversely, the absence of such terms apparently has an unfavorable effect on the internal structure of the system, and its investigation becomes rather difficult; at least, we were unable to carry it out in the latter case. The additional terms mentioned were chosen chiefly with the aim of obtaining a Fredholm system that is readily amenable to investigation and solvable.*

Meanwhile, as is evident from (7), they also introduce a simplification into the kernels of the system, from which the terms with logarithmic singularities then drop out. If, however, the named terms are removed from expression (4) for $\psi(z)$, then, in addition to the essential difficulties noted, terms containing a logarithm will appear in the kernels of the system, which will to a known extent complicate numerical calculations.

§ 2. We shall now turn to the proof of solvability of system (6). Suppose that the homogeneous system ($f_1 = f_2 = 0$) has some nontrivial solution $\nu_1^{(0)}(s)$

and $\nu_2^{(0)}(s)$; the functions (4) corresponding to it will be denoted by $\varphi^{(0)}(z)$ and $\psi^{(0)}(z)$. By the uniqueness theorem, everywhere in the domain S we have $\varphi^{(0)}(z) = ikz + C,\ \psi^{(0)}(z) = 0$, where k is real and $C = C_1 + iC_2$ is a complex constant. Writing down for this case the condition that the normal component of the displacement vector vanish on L, and integrating it along the arc s, we arrive at the equality $k(\xi^2+\eta^2)+2C_2\xi-2C_1\eta = C_0$, where C_0 is a real constant. Suppose for the moment that the contour L is not a circle; then from the last equality and then from the first formula (4) (putting $z = 0$ in it) we successively find

$$k = C_1 = C_2 = C_0 = 0, \qquad \int_L [\nu_1^{(0)}(s)\dot{t}\ddot{t} - \nu_2^{(0)}(s)]\, dt = 0. \tag{8}$$

Taking into account the last of conditions (8) and carrying out obvious transformations, we shall have on the contour L

$$\nu_1^{(0)}(s)\dot{t}\ddot{t} - \nu_2^{(0)}(s) = \omega'(t), \qquad \omega = \omega_1 + i\omega_2, \qquad \chi = \chi_1 + i\chi_2, \qquad A = A_1 + iA_2,$$

$$-\varkappa\omega(\bar{t}) + \nu_1^{(0)}(s)[\bar{t}\,\dot{t}\,\ddot{t} + (\varkappa - 1)\dot{\bar{t}}] + \nu_2^{(0)}(s)\dot{\bar{t}} = \chi(t) - A, \tag{9}$$

* Representation (4), at least in its essential part (we do not include in it the terms mentioned above, entering into $\psi(z)$), can be obtained by directly applying the method proposed in papers ([3], [4]).

where $\omega(z)$ and $\chi(z)$ are functions regular outside L and equal to zero at infinity; A is a functional depending on $\psi_1^{(0)}(s)$ and $\psi_2^{(0)}(s)$, and its value is not difficult to write down*. We now multiply the second of equalities (9) by $\dot{t}$ and separate in it the real and imaginary parts; into the two real relations obtained we substitute the expressions for $\psi_j^{(0)}(s)$ from the first equality (9). After this we arrive at formulas identically coinciding in form with conditions (1) and (2); in them, instead of the functions u, v and p, q, there appear respectively ω_1, ω_2 and χ_1, χ_2, and, moreover, $2\mu v_n = A_1\eta + A_2\xi$ and $T = 0$. Thus the functions $\omega(z)$ and $\chi(z)$ give a solution of the problem of elasticity theory for the infinite domain exterior to S, under the indicated special conditions on L; for them the equality holds

$$\int_L (v_n N + v_\tau T)\, ds = \frac{1}{2\mu} \int_L (A_1 X_n - A_2 Y_n)\, ds = 0,$$

where v_τ is the tangential component of displacement, N is the normal stress, and X_n, Y_n are the components along the axes of Cartesian coordinates of the acting external forces; their principal vector, by virtue of the uniqueness of $\omega(z)$ and $\chi(z)$, is equal to zero. Hence it follows at once that the functions

$\omega(z)$ and $\chi(z)$ (which, by assumption, vanish at infinity) are identically equal to zero everywhere outside L; in turn, this entails the required equalities** $\psi_1^{(0)} = \psi_2^{(0)} = 0$; from these it also follows that $A_1 = A_2 = 0$. Thus the nonhomogeneous system (6) always has a unique solution.

One can construct a system of integral equations, slightly modified in comparison with (6) and at the same time solvable (for any right-hand side), for the case when the contour L is a circle. To achieve this, it is sufficient to add to the left-hand side of boundary condition (2) the elementary operator $S^{-1}\operatorname{Re} it\dot{\bar{t}}\operatorname{Im}\varphi'(0)$, taking the functional $\varphi'(0)$ from (4). Indeed, the integral of the stress component T, taken along the circumference L, in this case expresses the principal moment of the external forces applied to the boundary, and therefore must be equal to zero. Consequently, any solution of the new system necessarily sends the added operator to zero and, therefore, coincides with the solution of the former system, subject to the additional condition $\operatorname{Im}\varphi'(0) = 0$. In the rest, the proof of solvability proceeds as above. Incidentally, it can be shown directly that system (6) is also solvable for a circle L, provided only that the prescribed tangential stress satisfies the mentioned obligatory condition***.

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CITED LITERATURE

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* We find A by multiplying, term by term, both sides of the second formula (9) by $t^{-1}\,dt$ and integrating over the contour L.

** Of course, they remain valid also for a contour containing rectilinear portions having conjugation with adjacent arcs; on these portions $\dot\xi\ddot\eta - \dot\eta\ddot\xi = 0$, and therefore from the first formula (9) one cannot immediately extract the equality $\psi_1^{(0)}(s)$ everywhere on L; however, it can be established from the imaginary part of the second formula (9), previously multiplied by $\dot t$.

*** Without detriment to the completeness of the investigation, one may exclude from consideration the case when the domain occupied by the medium is a disk, since for this case there exists a fairly simple solution (in quadratures); it was found in due course by N. I. Muskhelishvili ([2]).

Note: Figure translations are in progress. See original paper for figures.