MATHEMATICS

A. D. SOLOV' EV

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. D. SOLOV' EV

DETERMINATION OF THE CLASS OF CONVERGENCE OF AN INTERPOLATION SERIES FOR CERTAIN PROBLEMS

(Presented by Academician A. N. Kolmogorov on 12 X 1956)

Let an entire function be given,

$$\Phi(z) = \sum_{n=0}^{\infty} \frac{z^n}{m_n},$$

where the numbers m_n satisfy the conditions: $m_n \neq 0$ and $|m_{n+1}/m_n|$ tends monotonically to infinity. By the class $[\Phi, \sigma]$ we shall mean the class of entire functions

$$F(z) = \sum_{k=0}^{\infty} c_k z^k,$$

for which

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n m_n|} < \sigma < \infty.$$

It is not difficult to show that for functions of this class

$$\overline{\lim_{r\to\infty}} \frac{m(r)}{m^{\Phi}(\sigma r)} = 0,$$

where

$$m(r) = \ln \max_{|z|=r} |F(z)|, \qquad m^{\Phi}(r) = \ln \max_{|z|=r} |\Phi(z)|.$$

The function

$$f^{\Phi}(\zeta) = \sum_{n=0}^{\infty} \frac{c_n m_n}{\zeta^{n+1}}$$

will be called Φ-associated with the function $F(z)$. For entire functions of the class $[\Phi, \sigma]$, all singularities of the Φ-associated functions lie inside the disk $|\zeta| < \sigma$.

Consider the sequence of linear functionals

$$A_n = A_n[F] = \frac{1}{2\pi i} \int_C \zeta^n \varphi_n^\Phi(\zeta) f^\Phi(\zeta)\, d\zeta, \qquad n = 0, 1, 2, \dots, \tag{1}$$

where

$$\varphi_n^\Phi(\zeta) = \sum_{k=0}^{\infty} \frac{a_{nk}}{m_{n+k}} \zeta^k \quad (a_{n0} = 1)$$

are regular for $|z| < R$; $F(z) \in [\Phi, R]$; the contour C encloses all singularities of $f^\Phi(\zeta)$ and lies in the disk $|z| < R$. Functionals (1) that are invariant with respect to the function $\Phi(z)$ will be called moments.

Let us write the formal identity

$$\Phi(z\zeta) = \sum_{n=0}^{\infty} p_n(z)\, \zeta^n \varphi_n^\Phi(\zeta).$$

Comparing the coefficients of equal powers of ζ, we obtain finite recurrence relations for $p_n(z)$, from which these functions are determined successively and uniquely. It is easy to show that $p_n(z)$ is a polynomial of degree n, invariant with respect to the function $\Phi(z)$. The functions $p_n(z)$ are called interpolation polynomials, and the series

$$\sum_{n=0}^{\infty} A_n p_n(z) \sim F(z) \tag{2}$$

the interpolation series of the function $F(z)$.

We shall say that the class $[\Phi, \sigma]$ is the exact convergence class of the interpolation series if, for every entire function $F(z) \in \mathfrak{C}[\Phi, \sigma]$, the series (2) converges to $F(z)$ uniformly in every finite disk, and if for every $\varepsilon > 0$ there exists $F_1(z) \in [\Phi, \sigma + \varepsilon]$ for which the series (2) diverges at least at one point. Denoting the exact convergence class by K_0, we shall record this fact as follows: $K_0 = [\Phi, \sigma]$.

The following assertion is true (²):

Theorem 1. *Let the moments (1) be given, and suppose that*

$$\lim_{n\to\infty} m_n \varphi_n^\Phi(\zeta) = \varphi(\zeta)$$

uniformly in every disk $|\zeta| \leq r < R_1 \leq R$.

If $\varphi(\zeta)$ *has zeros inside the disk* $|\zeta| < R_1$, *then* $K_0 = [\Phi, |\alpha_1|]$, *where* α_1 *is the zero of the function* $\varphi(\zeta)$ *nearest to the origin.*

A generalization of this theorem is Theorem 2.

Theorem 2. *Let the moments (1) be given, and suppose that*

$$\lim_{n\to\infty} m_{nk+s}\varphi^{\Phi}_{nk+s}(\zeta) = \varphi_s(\zeta), \qquad s = 0, 1, 2, \dots, k-1,$$

uniformly in every disk $|\zeta| \leq r < R_1 \leq R$.

If the function

$$\Delta(\zeta) = \begin{vmatrix} \varphi_0(\zeta) & \varphi_1(\zeta) & \cdots & \varphi_{k-1}(\zeta) \\ \varphi_0(\varepsilon\zeta) & \varepsilon\varphi_1(\varepsilon\zeta) & \cdots & \varepsilon^{k-1}\varphi_{k-1}(\zeta) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_0(\varepsilon^{k-1}\zeta) & \cdots & \cdots & \varepsilon^{(k-1)^2}\varphi_{k-1}(\varepsilon^{k-1}\zeta) \end{vmatrix},$$

where $\varepsilon = e^{2\pi i/k}$, *has zeros in the disk* $|\zeta| < R_1$, *then* $K_0 = [\Phi, |\alpha_1|]$, *where* α_1 *is the zero of the function* $\Delta(\zeta)$ *nearest to the origin.*

By virtue of the principle of duality (3), these theorems assert that the system $\{z^n \varphi^{\Phi}_n(z)\}$ forms a basis in the disk $|z| < |\alpha_1|$ and does not form one in any larger disk.

We shall prove a simple assertion that makes it possible to apply these theorems in a number of interesting problems.

Theorem 3. *Let the moments (1) be given, and let* $\{\lambda_n\}$ *be a sequence of complex numbers satisfying the conditions:*

$$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = q, \qquad |\lambda_n| \leq |\lambda_{n+1}|, \qquad n = 0, 1, 2, \dots.$$

If

$$\lim_{n\to\infty} m_n \varphi^{\Phi}_n\left(\frac{\zeta}{\lambda_n}\right) = \widetilde{\varphi}(\zeta) = \sum_{k=0}^{\infty} a_k \zeta^k$$

uniformly in every disk $|\zeta| \leq r < R_1$, *then*

$$\lim_{n\to\infty} m'_n \varphi^{\Psi}_n(\zeta) = \varphi(\zeta) = \sum_{k=0}^{\infty} a_k q^{k(k-1)/2} \zeta^k$$

uniformly in every disk $|\zeta| \leq r < R_1$, *where*

$$\Psi(z) = \sum_{n=0}^{\infty} \frac{z^n}{m'_n},$$

$$m'_n = m_n \lambda_1 \cdots \lambda_{n-1} \quad (m'_0 = m_0,\ m'_1 = m_1).$$

In other words, in this case $K_0 = [\Psi, |\alpha_1|]$, where α_1 is the zero of the function $\varphi(\zeta)$ closest to the origin ($|\alpha_1| < R_1$).

Proof. We have

$$m_n \varphi_n^\Phi\left(\frac{\zeta}{\lambda_n}\right) = \sum_{k=0}^{\infty} \frac{a_{nk} m_n}{m_{n+k} \lambda_n^k} \zeta^k,$$

$$m'_n \varphi_n^\Psi(\zeta) = \sum_{k=0}^{\infty} \frac{a_{nk} m_n}{m_{n+k} \lambda_n \cdots \lambda_{n+k-1}} \zeta^k.$$

Next:

$$\lim_{n\to\infty} \frac{a_{nk} m_n}{m_{n+k} \lambda_n \cdots \lambda_{n+k-1}} = a_{kq}^{k(k-1)/2}$$

and, moreover:

$$\left| \frac{a_{nk} m_n}{m_{n+k} \lambda_n \lambda_{n+1} \cdots \lambda_{n+k-1}} \right| \le \left| \frac{a_{nk} m_n}{m_{n+k} \lambda_n^k} \right|,$$

whence the assertion of the theorem follows.

The assertion corresponding to Theorem 2 is formulated analogously. Let us consider several examples, using the notation of the last theorem.

1. **The Abel–Goncharov problem.**

$$\varphi_n^\Phi(\zeta) = \frac{1}{n!} e^{\lambda_n \zeta}, \qquad \Phi(\zeta) = e^\zeta, \qquad A_n[F] = \frac{F^{(n)}(\lambda_n)}{n!}.$$

Suppose that

$$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = q \ne 1, \qquad |\lambda_n| \le |\lambda_{n+1}|.$$

Then

$$K_0 = [\Psi, |\alpha_1|],$$

where

$$\Psi(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \lambda_1 \cdots \lambda_{n-1}}, \qquad \varphi(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{n!} z^n.$$

Consider special cases:

a)

$$\lambda_n = \nu_{ne}^{i\alpha(-1)^n}, \qquad \nu_n \le \nu_{n+1}, \qquad \frac{\nu_n}{\nu_{n+1}} \to 1,$$

$$\Psi(z) = \sum_{n=0}^{\infty} \frac{z^n}{n! \nu_1 \cdots \nu_{n-1}}, \qquad \Delta(z) = \cos(2z \sin \alpha).$$

Consequently,

$$K_0 = \left[\Psi, \frac{\pi}{4\sin\alpha}\right].$$

b)

$$\frac{\lambda_n}{\lambda_{n+1}} \to 0, \qquad \Psi(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\lambda_1 \cdots \lambda_{n-1}}, \qquad \varphi(z) = 1 + z, \qquad |\alpha_1| = 1.$$

Thus,

$$K_0 = [\Psi, 1].$$

2. A generalization of the Abel–Goncharov problem.

$$\varphi_n^{\Phi}(z) = \frac{1}{n!} f(\lambda_n z), \qquad \Phi(z) = e^z, \qquad f(z) = \sum_{k=0}^{\infty} a_{kz}^k, \qquad a_0 = 1.$$

Let again

$$\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = q, \qquad |\lambda_n| \leqslant |\lambda_{n+1}|.$$

Then

$$K_0 = [\Psi, |\alpha_1|],$$

where

$$\Psi(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\lambda_1 \ldots \lambda_{n-1}}, \qquad \varphi(z) = \sum_{k=0}^{\infty} a_k q^{k(k-1)/2} z^k.$$

3. On A. O. Gelfond’ s moment problem[1].

a)

$$\zeta^n \varphi_n^{\Phi}(\zeta) = \frac{u^n(\zeta) + v^n(\zeta)}{2n!}, \qquad \Phi(\zeta) = e^{\zeta},$$

$$u(\zeta) = \zeta + \alpha\zeta^{k+1} + \ldots, \qquad v(\zeta) = \zeta + \beta\zeta^{k+1} + \ldots, \qquad \alpha \neq \beta.$$

Then

$$K_0 = [\Psi, |\alpha_1|],$$

where

$$\Psi(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1+1/k}}, \qquad \varphi(z) = e^{\alpha z^k} + e^{\beta z^k}, \qquad |\alpha_1| = \left(\frac{\pi}{|\alpha - \beta|}\right)^{1/k}.$$

It follows from this that the system $\{u^n(\zeta) + v^n(\zeta)\}$, under the condition $\alpha \neq \beta$, cannot form a basis in a neighborhood of the origin. (We note that the condition $\alpha \neq \beta$ may be replaced by the condition $u(\zeta) \not\equiv v(\zeta)$.)

b)

$$\zeta^n \varphi_n^{\Phi}(\zeta) = \frac{u_{\alpha_n}^n(\zeta)}{n!}, \qquad \alpha_n = \begin{cases} 0, & n = 2k, \\ 1, & n = 2k+1; \end{cases} \qquad \Phi(\zeta) = e^{\zeta}.$$

$$u_0(\zeta) = \zeta + \alpha\zeta^{k+1} + \dots, \qquad u_1(\zeta) = \zeta + \beta\zeta^{k+1} + \dots, \qquad \alpha \neq \beta.$$

Then

$$K_0 = [\Psi, |\alpha_1|],$$

$$\Psi(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{1+1/k}}, \qquad \Delta(\zeta) = \operatorname{ch}(\alpha - \beta)\zeta^k, \qquad |\alpha_1| = \left(\frac{\pi}{2|\alpha - \beta|}\right)^{1/k}.$$

Moscow State University
named after M. V. Lomonosov

Received
12 X 1956

REFERENCES

[1] A. O. Gelfond, *Calculus of finite differences*, Moscow—Leningrad, 1952. [2] M. A. Evgrafov, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **17**, 421 (1953). [3] A. I. Markushevich, *Mat. sbornik*, **17** (59), 2, 211 (1945).