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Abstract

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ON PERIODIC SOLUTIONS IN A NEIGHBORHOOD OF A SINGULAR POINT OF A DYNAMICAL SYSTEM

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Below we consider systems

$$\ddot{x}_i + g_i(x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n) = 0 \quad (i = 1, \dots, n), \tag{1}$$

It is assumed that

$$g_i(-x_1, \dots, -x_n, y_1, \dots, y_n) = -g_i(x_1, \dots, x_n, y_1, \dots, y_n). \tag{2}$$

In view of (2), the origin is a stationary point of system (1).

The article studies the question of periodic solutions of system (1) with small amplitudes, the question of the dependence of amplitudes on periods. Systems obtained from (1) by a small perturbation are also considered.

1. In (1) it is indicated that to system (1) there may be associated an equation

$$\varphi = A(\varphi, T) \tag{3}$$

with an operator A, acting in a certain functional space of vector-functions and depending on a positive parameter T, whose solutions determine the periodic solutions of system (1) with period T.

General theorems of nonlinear functional analysis lead to diverse assertions about the location of sets of solutions of equation (3) in a neighborhood of zero in the functional space. On passing to other terms, these same assertions describe the location of sets of closed integral curves in a neighborhood of the stationary point of the $2n$-dimensional phase space of system (1).

The assertions given below may also be interpreted as theorems on periodic solutions of Lyapunov systems. The consideration of the systems is carried out under such assumptions as do not use knowledge of an analytic first integral, the knowledge of which is essential in Lyapunov’ s theory ([2], Chap. 4).

The theorems of the first sections follow from various theorems on bifurcation points established earlier by the author ([3]) outside connection with nonlinear mechanics. The subsequent theorems are consequences of propositions on the location of the spectrum of a nonlinear operator in a neighborhood of a bifurcation point ([3], Chap. 4) and of some theorems from ([4]).

2. We shall say that a set $\mathfrak{N}$ of periodic solutions of the autonomous system (1) has limiting period T, if the periods of solutions from the set $\mathfrak{N}$ tend to T when the amplitudes of these solutions tend to zero. Sets $\mathfrak{N}$ with limiting period T may consist of a countable or a continuum number of solutions.

By R^2 we denote the $2n$-dimensional phase Euclidean space with coordinate system $x_1, \dots, x_n,\ y_1, \dots, y_n$. To the periodic solutions of system (1) there correspond closed integral curves of the system.

$$\dot{x}_i = y_i,$$
$$\dot{y}_i = -g_i(x_1, \dots, x_n, y_1, \dots, y_n) \qquad (i = 1, \dots, n) \tag{4}$$

in the space R^{2n}. By $\mathfrak{N}^*$ we shall denote the set of points of the space R^{2n} lying on closed integral curves of system (4) corresponding to periodic solutions of system (1) from the collection $\mathfrak{N}$.

We shall say that the collection $\mathfrak{N}$ of periodic solutions with limiting period T forms a continuous branch in a neighborhood of the zero stationary point if the intersection of $\mathfrak{N}^*$ with the boundary of any neighborhood of the stationary point, lying wholly inside a ball of some radius determined by the collection $\mathfrak{N}$, is nonempty. Roughly speaking, $\mathfrak{N}$ is a continuous branch of periodic solutions if $\mathfrak{N}^*$ contains a two-dimensional manifold whose interior point is the stationary point.

By C below we denote the matrix of order n with elements

$$c_{ij} = \frac{\partial}{\partial x_j} g_i(0, \dots, 0, 0, \dots, 0) \qquad (i, j = 1, \dots, n). \tag{5}$$

Theorem 1. *Let $4\pi^2/T_k^2$ be an odd-multiple positive root of the characteristic equation of the matrix C. Let the sum of the multiplicities of the other positive roots of the characteristic equation whose integer multiples are $4\pi^2/T_k^2$ be even.*

Then to the root $4\pi^2/T_k^2$ there corresponds a continuous collection $\mathfrak{N}_k$ of periodic solutions of system (1) with limiting period T_k. The collection $\mathfrak{N}_k$ forms a continuous branch of periodic solutions in a neighborhood of the zero stationary point.

Theorem 2. *Let the functions g_i, in addition to condition (2) of oddness with respect to the first group of variables, satisfy the condition*

$$g_i(x_1, \dots, x_n, \dots, y_1, y_n) = g_i(x_1, \dots, x_n, y_1, \dots, y_n) \quad (6)$$
$$(i = 1, \dots, n)$$

of evenness with respect to the second group of variables. Let $4\pi^2/T_k^2$ *be an odd-multiple positive root of the characteristic equation of the matrix* C*. Let the sum of the multiplicities of the other positive roots of the characteristic equation whose integer odd multiples are* $4\pi^2/T_k^2$ *be even.*

Then the assertion of Theorem 1 is valid.

We note that in the case of a system with one degree of freedom the assertion of the theorem is obvious. In the case of systems with many degrees of freedom, Theorems 1 and 2 give, generally speaking, conditions for the existence in a neighborhood of the stationary point of several series of periodic solutions corresponding to different $4\pi^2/T_k^2$.

3. The assumption of odd multiplicity of $4\pi^2/T_k^2$ in Theorems 1 and 2 is essential. In the general case of roots of even multiplicity, the investigation becomes more complicated and leads to conditions for the existence of periodic solutions whose formulation uses the terms following the linear ones in the Taylor expansions of the functions g. The systems more simply investigated are

$$\ddot{x}_i + g_i(x_1, \dots, x_n) = 0 \qquad (i = 1, \dots, n), \quad (7)$$

if

$$g_i(x_1, \dots, x_n) = \frac{\partial}{\partial x_i} G(x_1, \dots, x_n) \qquad (i = 1, \dots, n). \quad (8)$$

Theorem 3. *Let the functions* g_i *satisfy conditions (2) and (8).*

Then to each positive root $2\pi^2/T_k^2$ *(of any multiplicity) of the characteristic equation of the matrix* C *there corresponds*

the set $\mathfrak{R}_k$ of periodic solutions of system (7) with limiting period T_k.

We have not been able to determine whether, under the conditions of Theorem 3, $\mathfrak{R}_k$ always forms a continuous branch of periodic solutions.

4. Denote by L_k the subspace in R^{2n} consisting of the eigenvectors of the matrix
$$\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix},$$
corresponding to the eigenvalue $4\pi^2/T_k^2$.

Theorem 4. $4\pi^2/T_k^2$, under the conditions of Theorem 1 (Theorem 2 or Theorem 3), is not an integral multiple (an odd integral multiple) of other positive roots of the characteristic equation of the matrix C.

Then the set of points $\mathfrak{R}_k^{**}$ at zero of the phase space R^{2n} is tangent to L_k in the sense that

$$\lim_{\bar{x}\in\mathfrak{R}_k^{**},\, \|\bar{x}\|\to 0} \frac{1}{\|\bar{x}\|}\rho(\bar{x}, L_k) = 0, \tag{9}$$

where

$$\|\bar{x}\| = (x_1^2 + \dots + x_n^2 + y_1^2 + \dots + y_n^2)^{1/2}.$$

5. Distinct periodic solutions of system (1) or (7) generally have different periods. The question of the dependence of amplitudes on periods is of interest (see ([5],[8]) and others). Topological considerations lead to simple assertions analogous, for example, to those indicated for the well-known Duffing equation in ([6]) (pp. 88-94).

Let $4\pi^2/T_k^2$ be a simple root of the characteristic equation of the matrix C, to which the eigenvector $\{\xi_1, \dots, \xi_n\}$ corresponds. Suppose that the functions g_i admit the representation

$$g_i = \sum_{j=1}^{n} c_{ij}x_j + P_i(x_1, \dots, x_n, y_1, \dots, y_n) + \omega_i \quad (i = 1, \dots, n), \tag{10}$$

where all P_i are homogeneous forms of some order s (for example, the third), and ω_i are terms of higher order of smallness:

$$\lim_{\|\bar{x}\|\to 0} \frac{1}{\|\bar{x}\|^s} \sum_{i=1}^{n} |\omega_i| = 0. \tag{11}$$

All further considerations are carried out under the assumption that

$$\gamma = \int_{-\pi}^{\pi} P(t)\sin t\, dt \neq 0, \tag{12}$$

where

$$P(t) = \sum_{i=1}^{n} \xi_i P_i(\xi_1 \sin t, \dots, \xi_n \sin t, \xi_1 \cos t, \dots, \xi_n \cos t). \tag{13}$$

Theorem 5. Suppose condition (2) is satisfied. Let the simple root $4\pi^2/T_k^2$ of the characteristic equation of the matrix C not be an integral multiple of other roots of the characteristic equation. Let $\mathfrak{R}_k$ be the set of periodic solutions of system (1) with limiting period T_k.

Then, for $\gamma < 0$, the amplitudes of the periodic solutions from $\mathfrak{R}_k$ increase as the period increases. For $\gamma > 0$, the amplitudes increase as the period decreases.

Theorem 6. *Suppose that conditions (2) and (6) are satisfied. Suppose that the simple root $4\pi^2/T_k^2$ of the characteristic equation of the matrix C is not an odd multiple of other roots of the characteristic equation.*

Then the assertions of Theorem 5 are valid.

Theorems 5 and 6 mean that the condition $\gamma < 0$ corresponds to a soft restoring force, while $\gamma > 0$ corresponds to a hard one ([6], p. 92).

6. Let us now consider the system

$$\ddot{x}_i + g_i(x_1, \dots, x_n, \dot{y}_1, \dots, \dot{y}_n) + \varepsilon^2 f_i(t, x_1, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n) = 0 \quad (i = 1, \dots, n), \tag{14}$$

where the f_i are periodic in t with some period T and satisfy the condition

$$f_i(-t, -x_1, \dots, -x_n, y_1, \dots, y_n) = -f_i(t, x_1, \dots, x_n, y_1, \dots, y_n), \tag{15}$$

and ε^2 is a positive small parameter. We shall regard the period T as close to T_k. We shall assume that condition (12) and the condition

$$a = \int_{-\pi}^{\pi} F(t) \sin t \, dt \neq 0, \tag{16}$$

are satisfied, where

$$F(t) = \sum_{i=1}^{n} \xi_i f_i \left(\frac{T}{2\pi} t, 0, \dots, 0 \right). \tag{17}$$

Each periodic solution $\{x_i(t)\}$ of system (1) belongs to the obvious continuous family of solutions $\{x_i(t + t_0)\}$, obtained from the original periodic solution by phase shifts. The nonautonomous system (14) may already have isolated periodic solutions. In some cases one can indicate an estimate for the number of such solutions.

The odd periodic solutions of system (1) under the conditions of Theorems 5 and 6 are fixed points of nonzero index ([3], Ch. 2) of certain completely continuous operators. The passage to the perturbed system (14) corresponds to small perturbations of these operators. Therefore the perturbed operators also preserve fixed points, which yield periodic solutions of system (14).

Theorem 7. *Suppose that, for system (1), the conditions of Theorem 5 are satisfied. Suppose that condition (16) is satisfied.*

Then, for $\gamma > 0$, system (14) has, for small ε, not fewer than three periodic solutions, if T is sufficiently close to T_k and $T > T_k$. For $\gamma < 0$, system (14) *has not fewer than three periodic solutions, if ε is small, T is sufficiently close to T_k, and $T < T_k$.*

Theorem 6 can also be supplemented by an analogous assertion. Theorem 7 is obtained with the aid of the general theorems of the paper ([4]). The amplitude curves under the conditions of Theorem 7 have the form analogous to the graphs given for the case of the Duffing equation in ([6]) (Fig. 40).

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Note: Figure translations are in progress. See original paper for figures.