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Abstract

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MATHEMATICS

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ON CONDITIONS NECESSARY AND SUFFICIENT FOR COMPLETE APPROXIMATIVE SOLVABILITY OF EQUATIONS OF A VERY GENERAL NATURE

(Presented by Academician S. L. Sobolev on 10 IX 1956)

1°. The basic idea of the well-known approximative method in the theory of integral equations, consisting in replacing a given kernel by one that is degenerate from the point of view of general approximative considerations, evidently consists in the following ([1]): if the left- and right-hand sides of the equation $L(y) = f(x)$ are approximated by operators $L_n(y)$ and, respectively, by functions $f_n(x)$, then in certain cases the solutions $\varphi_n(x)$ of the equations $L_n(y) = f_n(x)$ uniformly approximate the solutions of the equation $L(y) = f(x)$. In any case this holds when the Fredholm equation of the second kind has a unique solution.

However, in the case of an infinite set of solutions of the linear equation $L(y) = f(x)$, it does not at all follow that the conditions $L_n(y) \to L(y)$ and $f_n(x) \to f(x)$, even in the case of uniform convergence, entail the possibility of uniformly approximating all solutions of the equation $L(y) = f(x)$ by solutions of the equation $L_n(y) = f_n(x)$ for $n = 1, 2, ...$ The same circumstance occurs not only in the case of integral equations, but also in the corresponding approximative problem in finite-dimensional linear spaces.

In the work ([2]) a criterion was established that is sufficient in order that the conditions $L_n(y) \to L(y)$, for a **fixed** right-hand side $f(x)$, entail the possibility of uniformly approximating all solutions of the equation $L(y) = f(x)$ by solutions of the equations $L_n(y) = f(x)$.

If the right-hand side is not fixed and, consequently, one considers the more natural approximative problem, then it turns out that this same condition will be necessary and sufficient in order that the condition $L_n(y) \to L(y)$, with $f_n(x) \to f(x)$, entail the possibility of uniform approximation of all solutions of the equation $L(y) = f(x)$ by solutions of the equations $L_n(y) = f_n(x)$.

Let $L(p)$ and $L_n(p)$ be continuous linear and normalized (i.e. satisfying the condition $L(0) = 0$) mappings of a linear space R into R' or onto R^1. We shall

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agree to denote by the symbol $L_n \to L$ the uniform convergence of the mappings L_n to L in the domain $G \in R$.

Theorem 1. *In order that the conditions $L_n \to L$ for all $q_n \to q$, where $q_n, q \in R^1$, and L_n are linear mappings of R onto R', entail the possibility of approximation, by solutions of the equations $L_n(p) = q_n$ $(n = 1, 2, ...)$, of all solutions of the equation $L(p) = q$ lying in the domain $G \in R$, it is necessary and sufficient that the mappings of the sequence L_n satisfy the following condition:*

Whatever ε may be, there exists such a $\delta(\varepsilon)$, the same for all L_n, that whenever $\|q\| < \delta$, one can indicate at least one solution of the equation $L_n(p) = q$ satisfying the condition $\|p\| < \varepsilon$; in other words, for each ε there must exist $\delta(\varepsilon)$ such that the image of the ε-neighborhood of zero under all mappings L_n covers the δ-neighborhood of zero.

Proof. In paper ([3]) it was established that from the condition imposed on the system L_n at zero, by virtue of the linearity of the mappings, the same condition follows at every point $p \in R$, i.e., for every p and any ε there is a $\delta(p, \varepsilon)$, independent of L_n, such that the image of the ε-neighborhood of the point p covers the δ-neighborhood of the point $L_n(p)$.

A system of mappings satisfying such a condition at each point $p \in R$ will be called a system of **uniformly open mappings**.* In its logical structure this concept is, obviously, close to the concept of an equicontinuous system of functions, but uniformity holds only with respect to the mappings, and not with respect to $p \in R$. The validity of our theorem easily follows from a more general theorem.

Theorem 2. *Let a sequence of continuous mappings f_n of the space R into R^1 converge uniformly to a mapping f of R into R^1. In order that the sequence $f(y_n)$ of complete preimages of the points y_n of any convergent sequence $y_n \in R'$ converge to $f^{-1}(y)$, $y = \lim y_n$, it is necessary and sufficient that the f_n form a system of uniformly open mappings.*

Necessity. Assuming that the f_n do not form a system of uniformly open mappings, we immediately find such an $x \in R$ and ε that $f_n(u_\varepsilon(x))$ does not cover the corresponding δ-neighborhood of $f_n(x)$. Hence, there will be a sequence $y_n \subset R^1$ such that $y_n \to y = f(x)$, y_n does not belong to $f_n(u_\varepsilon(x))$, i.e. $\mathrm{lt}\, f^{-1}(y_n) \neq f^{-1}(y)$.

The **proof of the sufficiency** of the condition differs little in idea from the proof of the corresponding assertion of Theorem 2 of paper ([2]), and therefore is not given here.

By virtue of the remark made above, the f_n form a system of uniformly open mappings, and Theorem 2 can be applied.

It is clear that the assertion of Theorem 2 on the convergence of complete preimages is equivalent in substance to the possibility of approximating all solutions of the equation $L(p) = q$ by solutions of the system of equations $L_n(p) = q_n$.

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2°. For nonlinear equations, Theorem 2 obviously establishes the following proposition.

If nonlinear mappings $R_n(p)$ uniformly approximate the mapping $R(p)$ and $q_n \to q$, then in order that all solutions of the equation $R(p) = q$, and only them, could be regarded as limits of all possible convergent sequences $\{p_n\}$, where p_n is a solution of the equation $R_n(p) = q_n$, it is necessary and sufficient that $R_n(p)$ constitute a system of **uniformly open** mappings.

It turns out that in order that all solutions of the equation $R(p) = q$ could be regarded as limits of solutions of the equations $R_n(p) = q_n$, without requiring that all such limits be solutions of the equation $R(p) = q$, it is enough to require nonuniform convergence of the mappings $R_n(p)$.

We shall first express these considerations for nonlinear integral equations.

Let the integral equation

$$\varphi(x) - \int_a^b K(x, t, \varphi(t))\, dt = f(x) \tag{1}$$

have a kernel $K(x, t, \varphi)$ continuous with respect to the aggregate (x, t, φ). We pose the problem of finding all solutions of this equation by means of the following approximative process.

* Whylburn, in defining a system of uniformly open mappings, requires uniformity with respect to $p \in R$.

Let functions $K_n(x, t, \varphi)$ be defined, also continuous with respect to the collection of variables (x, t, φ), and such that $K_n(x, t, \varphi)$, for each fixed function $\varphi(x)$, converge uniformly in the square $[a, b]$ to the function $K(x, t, \varphi(t))$ as $n \to \infty$. Moreover, let a sequence of continuous functions $f_n(x)$ converge uniformly to the function $f(x)$. Suppose that for the equations

$$\varphi(x) - \int_a^b K_n(x, t, \varphi(t))\, dt = f_n(x), \qquad n = 1, 2, \dots, \tag{2}$$

all continuous solutions are known. It is required to find necessary and sufficient conditions in order that, under such a process of approximation, all solutions of equation (1) lying among continuous functions could be regarded as limits of uniformly convergent sequences $\{\varphi_n(x)\}$, where each $\varphi_n(x)$ is a solution of the equation

$$\varphi(x) - \int_a^b K_n(x, t, \varphi(t))\, dt = f_n(x).$$

Theorem 3. Let $K(x, t, \varphi)$ and $K_n(x, t, \varphi)$ be expressions continuous with respect to the collection of variables (x, t, φ). Suppose further that $K_n(x, t, \varphi)$,

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for each fixed $\varphi(x)$, converge uniformly to $K(x,t,\varphi(x))$ in the square $[a,b]$. In order that this condition entail the possibility of representing all continuous solutions of the equation

$$\varphi(x) - \int_a^b K(x,t,\varphi(t))\,dt = f(x)$$

in the form of limits of uniformly convergent sequences, where each $\varphi_n(x)$ is a continuous solution of the equation

$$\varphi_n(x) - \int_a^b K_n(x,t,\varphi(t))\,dt = f_n(x),$$

and $f_n(x)$ is an arbitrary sequence of continuous functions (under the condition that for each n there exists at least one continuous solution of the equation), it is necessary and sufficient that the following condition be fulfilled:

For every continuous function $\varphi(x)$ and every ε, there must be found a δ, depending only on φ and ε, but not depending on n, such that for every n and every function $f(x)$ satisfying the condition $\|f(x) - F(x)\| < \delta$, where $F(x)$ is the value of the operator

$$\Omega(\varphi(x)) = \varphi(x) - \int_a^b K_n(x,t,\varphi(t))\,dt$$

for the given function $\varphi(x)$, there is at least one continuous solution $\psi(x)$ of the equation

$$\varphi(x) - \int_a^b K_n(x,t,\varphi(t))\,dt = f(x),$$

satisfying the condition $\|\psi(x) - F(x)\| < \varepsilon$.

Proof. It is obvious that, for a fixed $\varphi(x)$ and any $\eta > 0$,

$$\left| \int_a^b K(x,t,\varphi(t))\,dt - \int_a^b K_n(x,t,\varphi(t))\,dt \right| \leq$$

$$\leq \int_a^b |K(x,t,\varphi(t)) - K_n(x,t,\varphi(t))|\;dt < \eta,$$

starting from some $n(\eta)$, for all x and t. Hence the operators

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$$\Omega_n \equiv \varphi(x) - \int_a^b K_n(x, t, \varphi(t))\, dt$$

converge to the operator

$$\Omega(\varphi(x)) \equiv \varphi(x) - \int_a^b K(x, t, \varphi(t))\, dt$$

on the space C.

The validity of our theorem follows easily from the following assertion, which is an inessential generalization of Theorem 2.

Theorem 4. *Let a sequence of continuous mappings f_n of the space R into R' converge (generally speaking, nonuniformly) to a mapping f of R into R', and let $y_n \to y$, $y_n, y \in R'$.*

In order that all points of $f^{-1}(y)$ can be regarded as limits of sequences $\{x_n\}$, where each x_n belongs to $f_n^{-1}(y_n)$, it is necessary and sufficient that the f_n form a system of uniformly open mappings.

The proof is almost no different from the proof of the corresponding assertion in Theorem 2 and therefore is not given here.

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CITED LITERATURE

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[2] A. I. Polak, *Uspekhi Mat. Nauk*, **10**, 2 (64) (1955).
[3] G. T. Whyburn, *Ann. Soc. Polon. Math.*, **21** (1948).

Note: Figure translations are in progress. See original paper for figures.

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