ON THE OSCILLATION OF BODIES FLOATING IN A BASIN OF LIMITED DIMENSIONS

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Abstract

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HYDROMECHANICS

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ON THE OSCILLATION OF BODIES FLOATING IN A BASIN OF LIMITED DIMENSIONS

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1. Within the framework of the linear theory of waves, the problem is considered of the oscillation, about the equilibrium position, of a system of n bodies floating on the surface of a bounded volume of liquid.*

Theorem 1. *Small coupled oscillations of a heavy liquid in a bounded basin and of n bodies floating on its surface are described by the following system of equations:*

a) *equations of the quantity of motion*

$$M_k \frac{d^2 x_s^{(k)}}{dt^2} + \sum_{r=1}^{6} \sum_{m=1}^{n} m_{sr}^{(km)} \frac{d^2 x_r^{(m)}}{dt^2} + \int_S \gamma_s^{(k)}(P)\, \frac{d^2 \zeta(P,t)}{dt^2} + \mu_k^2 x_3^{(k)} \delta_{3s} = Q_s^{(k)},$$

$$k = 1, 2, \dots, n, \qquad s = 1, 2, 3;$$

b) *equations of moments*

$$I_s^{(k)} \frac{d^2 x_s^{(k)}}{dt^2} + \sum_{r=1}^{6} \sum_{m=1}^{n} m_{sr}^{(km)} \frac{d^2 x_r^{(m)}}{dt^2} + \int_S \gamma_s^{(k)}(P)\, \frac{d^2 \zeta(P,t)}{dt^2}\, dP + \sum a_{is}^{(k)} x_i^{(k)} = Q_s^{(k)}, \tag{1}$$

$$k = 1, 2, \dots, n, \qquad s = 4, 5, 6;$$

c) *the equation of constancy of pressure on the free surface*

$$\rho \int_S H(P,Q)\, \frac{d^2 \zeta(Q,t)}{dt^2}\, dQ + \sum_{i=1}^{6} \sum_{j=1}^{n} \frac{d^2 x_i^{(j)}}{dt^2} \gamma_i^{(j)}(P) + \rho g \zeta(P,t) = 0.$$

Here the following notation is adopted: M_k is the mass of the k-th body; $I_s^{(k)}$ is the moment of inertia of the k-th body relative to the s-th axis; $x_s^{(k)}$ are the coordinates of the center of gravity of the k-th body ($k = 1, 2, \dots, n;\ s = 1, 2, 3$) in the coordinate system x_1, x_2, x_3, associated with the free surface in the state of rest (the plane figure S); $x_s^{(k)}$ are the angular coordinates of the k-th body ($k = 1, 2, \dots, n;\ s = 4, 5, 6$); $m_{sr}^{(km)}$ are the added masses ($k, m = 1, 2, \dots, n;\ s, r = 1, 2, \dots, 6$):

$$m_{sr}^{(km)} = \rho \int_{\sigma_k} \Phi_r^{(m)} \frac{\partial \Phi_s^{(k)}}{\partial n}\, d\sigma;$$

ρ is the density; σ_k is the wetted surface of the k-th body (in the equilibrium position); $\Phi_r^{(m)}$ ($m = 1, 2, \dots, 6$) is the solution of the following Neumann problem:

* A special case of this problem was considered by Z. A. Perzhnyanko ([1]).

$$\frac{\partial \Phi_r^{(m)}}{\partial n} = 0 \quad \text{at the points } \Sigma + S + \sigma_1 + \dots + \sigma_{m-1} + \sigma_{m+1} + \dots + \sigma_n;$$

$$\frac{\partial \Phi_r^{(m)}}{\partial n} = \alpha_r^{(m)} \quad \text{at the points } \sigma_m, \qquad \text{if } r = 1, 2, 3;$$

$$\frac{\partial \Phi_r^{(m)}}{\partial n} = (\mathbf{r} \times \mathbf{n}_m^0) \cdot \mathbf{x}_{r-3}^0, \qquad \text{if } r = 4, 5, 6;$$

Σ is the surface of the reservoir; $\mathbf{r}$ is the radius vector of the points of σ_m; $\mathbf{n}_m^0$ is the unit vector of the external normal to σ_m; $\mathbf{x}_s^0$ is the unit vector of the s-th axis; $\alpha_r^{(m)}$ are the direction cosines of the vector $\mathbf{n}_m^0$;

$$\gamma_s^{(k)} = \rho \int_{\sigma_k} H(P, Q) \frac{\partial \Phi_s^{(k)}(Q)}{\partial n}\, dQ;$$

$H(P, Q)$ is the Green' s function of the Neumann problem for the domain τ, bounded by the surface $\Sigma + S + \sum_{k=1}^{n} \sigma_k$; $y = \zeta(P, t)$ is the equation of the free surface of the liquid; $\mu_k^2 x_3^{(k)}$ is the resultant of the Archimedean force and gravity applied to the k-th body; $Q_s^{(k)}$ are the external forces and moments (with the exception of the force of gravity); $\sum_{s=3}^{6} a_{is}^{(k)} x_i^{(k)}$ is the i-th component of the total moment of the Archimedean and gravity forces applied to the k-th body; g is the acceleration of gravity.

2. It is more convenient to transform system (1). Put $\zeta \equiv 0$ and $Q_s^{(k)} = 0$; then system (1) will be an ordinary conservative system. Denote by Y_s the normal coordinates of this system and take them as new generalized coordinates; then system (1) can be reduced to the form

$$\frac{d^2 Y_s}{dt^2} + \int_S \nu_s(P) \frac{d^2\zeta}{dt^2}\, dP + h_s^2 Y_s = L_s, \tag{2}$$

$$\rho \int_S H(P,Q) \frac{d^2\zeta}{dt^2}\, dQ + \sum_{i=1}^{6n} \frac{d^2 Y_i}{dt^2} \nu_i(P) + \rho g \zeta = 0, \qquad s = 1, 2, \dots, 6n.$$

The functions ν_s are expressed linearly through γ_k; h_s are the frequencies of the principal oscillations of the system of n bodies when $\zeta \equiv 0$. In these equations the functions $\nu_i(P)$ do not depend on the motion and are determined only by the geometry of the volume τ.

Let $L_s = 0$; then the periodic solutions of system (2) of the form

$$Y_s^{(k)} = q_s^{(k)} e^{i\omega_k t}, \qquad \zeta^{(k)} = z^{(k)} e^{i\omega_k t},$$

will be called principal oscillations. The numbers q_i and the functions $z^{(k)}$ satisfy the system

$$\rho g z - \omega^2 \left(\rho \int_S H(P,Q) z(Q)\, dQ + \sum_{i=1}^{6n} \nu_i(P) q_i \right) = 0;$$

$$h_s^2 q_s - \omega^2 \left(q_s + \int_S \nu_s(Q) z(Q)\, dQ \right) = 0, \qquad s = 1, 2, \dots, 6n. \tag{3}$$

The systems of equations (2) and (3) are analogous to those considered in papers ([2],[4]). Therefore the theorems proved in those papers extend to the case of the motions studied here.

3. **Theorem 2.** *When a system of n bodies moves in a bounded volume of fluid about an equilibrium position, there exist principal oscillations (periodic solutions of system (2)). The frequencies ω_n of these oscillations are real and form an infinite sequence such that*

$$\omega_n^2 \xrightarrow[n\to\infty]{} \infty.$$

Theorem 3. *For the stability of the periodic solutions of system (2), it is necessary and sufficient that*

$$h_s^2 > 0 \quad (s = 1, 2, \dots, 6n).$$

Theorem 4 (principle of superposition). *The system of principal oscillations is complete.*

Completeness in Theorem 4 is understood in the following sense. Let x be a vector having projections $x_0 = q_1$, $x_2 = q_2, \dots,$ $x_{6n} = q_{6n}$, $x_{6n+1} = z$, and let $x^{(k)}\{q_1^{(k)}, \dots, q_{6n}^{(k)}, z^{(k)}\}$ be a solution of system (3) corresponding to the k-th eigenfrequency $\omega = \omega_k$; then

$$x = \sum_{k=1}^{\infty} d_k x^{(k)},$$

where

$$d_k = \sum_{s=1}^{6n} h_s^2 q_s q_s^{(k)} + \rho g \int_S z(P) z^{(k)}(P)\, dP.$$

Theorem 5. *Under the condition $h_s^2 > 0$ $(s = 1, 2, \dots, 6n)$, the problem of determining the motion from initial conditions (the Cauchy problem for system (2)) has a unique solution, defined for every t, if $L_s(t)$ are functions of bounded variation on every finite interval of variation of t.* The solution of the Cauchy problem can be represented by the series

$$Y_i = \sum_{k=1}^{\infty} d_k q_i^{(k)} \cos \omega_k t + \sum_{k=1}^{\infty} d_k^* q_i^{(k)} \sin \omega_k t,$$

$$\zeta = \sum_{k=1}^{\infty} d_k z^{(k)} \cos \omega_k t + \sum_{k=1}^{\infty} d_k^* z^{(k)} \sin \omega_k t,$$

where

$$d_k = \sum_{s=1}^{6n} h_s^2 q_{s0} q_s^{(k)} + \rho g \int_S \zeta_0 z^{(k)}\, dP;$$

$$d_k^* = \frac{1}{\omega_k} \sum_{s=1}^{6n} h_s^2 q_{s0}^* q_s^{(k)} + \rho g \int_S \zeta_1 z^{(k)}\, dP;$$

$\{q_{10} \dots q_{6n0}, \zeta_0\}$ denote the initial position of the system; $\{q_{10}^* \dots q_{6n0}^*, \zeta_1\}$ denote the initial velocities.

4. The problem of determining the numbers ω_k^2 reduces to determining the roots of the meromorphic function

$$\Delta(\omega^2) = \left| \delta_{ij}(h_i^2 - \omega^2) + \frac{\omega^4}{\rho g} \sum_{k=1}^{\infty} \frac{\sigma_k^2 \nu_{ik} \nu_{jk}}{\omega^2 - \sigma_k^2} \right|, \tag{4}$$

where δ_{ij} is the Kronecker symbol; ν_{ik} are the Fourier coefficients of the function $\nu_i(P)$:

$$\nu_{ik} = \int_S \nu_i(P) \varphi_k(P) \, dP;$$

φ_k $(k = 1, 2, ...)$ are the normalized eigenfunctions

of the kernel function $H(P, Q)$; σ_k are the poles of the resolvent of the kernel $H(P, Q)$ (their physical meaning is the frequencies of free oscillations of the fluid, if all n bodies are rigidly fixed).

Various approximate and graphical methods may be developed for determining the roots of the equation $\Delta(\omega^2) = 0$, analogous to those developed in [3]. In [2] an equation similar to equation (4) was considered for the case of one degree of freedom of the rigid body, and the asymptotics of its roots was studied. In the present case analogous estimates can be obtained for any n. For this purpose the following theorem proves important.

Theorem 6. *The meromorphic function $\Delta(\omega^2)$ has only simple poles.*

5. The system of equations (1) can be used to study various properties of impact theory. We give here only one result. Suppose that only two bodies are floating on the surface of the fluid. It is always possible to choose the moment of inertia of one of the bodies and to strike it in such a way that it remains at rest, while the other body acquires a certain velocity, and not only translational but also rotational. Moreover, there exists an entire cone of such possible directions of impact. Corresponding to it are two cones of possible translational and rotational motions of the second body.

The results obtained may prove useful in constructing a hydrodynamic theory of the rolling of ships in a bounded water area (inside a harbor or in a canal).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.