ON A MEASURE OF HAUSDORFF TYPE

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Abstract

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MATHEMATICS

E. V. GLIVENKO

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A considerable class of measures for sets situated in Euclidean space or in a metric space with a countable base is obtained by the following construction. Choose a countable system of open sets Γ_i such that their diameter $\alpha(\Gamma_i) \to 0$ as $i \to \infty$. Let the regions Γ_i cover the space in the sense of Vitali.* Assign to each of the regions Γ_i an arbitrary number $\alpha_i \geq 0$, which we shall call the elementary measure $\mathrm{mes}_{\mathrm{e}}\, \Gamma_i$ of the region Γ_i. Fix an arbitrary $\varepsilon > 0$. Cover the set M by regions $\Gamma_{i_1}, \Gamma_{i_2}, \ldots$ from the system $\{\Gamma_i\}$ such that

$$d(\Gamma_{i_j}) < \varepsilon \quad (j = 1, 2, \ldots).$$

Consider

$$\sum_{j=1}^{\infty} \alpha_{i_j}.$$

The greatest lower bound of such sums over all possible coverings of the set M by regions from $\{\Gamma_i\}$ will be called the ε-measure $\mathrm{mes}_\varepsilon\, M$ of the set M. The limit of these measures as ε tends to zero will be called the measure of the set M of Hausdorff type and denoted by $\mathrm{mes}\, M$. A special case of such a measure is the ordinary linear Hausdorff measure in the plane, when the system $\{\Gamma_i\}$ consists of all disks with rational coordinates of their centers and rational radii, and the elementary measure of each disk is its diameter.

In order that the measure be convenient in use, it is necessary that it possess the so-called F-property, which consists in the following:

a) if the set M has finite measure, then for any $\varepsilon > 0$ there exists a closed set $F \subset M$ such that $|\mathrm{mes}\, M - \mathrm{mes}\, F| < \varepsilon$;

b) if the set M has infinite measure, then for any $N > 0$ there exists a closed set $F \subset M$ such that $\mathrm{mes}\, F > N$.

The F-property for the ordinary Hausdorff measure as applied to the sets $F_{\sigma\delta}$ was proved in 1952 by A. S. Besicovitch (^{1}). Earlier, R. A. Minlos (^{2}) proved the F-property for A-sets in the case of one special measure, but, in essence, his proof is applicable to the ordinary Hausdorff measure. In the case where the whole space is representable as the sum of a countable number of sets of finite measure, the F-property was also proved for measures of Hausdorff type (^{3}).

The present work is devoted to the proof of the F-property for measures of Hausdorff type in the general case as applied to A-sets.

Theorem 1. *Let, in a metric space with a countable base, some measure of Hausdorff type be introduced. Then:*

a) If some A-set M, situated in the space R, has finite measure, then, whatever $\varepsilon > 0$ may be, there exists a closed set $F \subset M$ such that $|\operatorname{mes} F - \operatorname{mes} M| < \varepsilon$.

b) If some A-set M has infinite measure, then, whatever the number N may be, there exists a closed set $F \subset M$ such that $\operatorname{mes} F > N$.

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* This means that for any point ξ of the space and for any number $\varepsilon > 0$ there exists an open set from the system $\{\Gamma_i\}$, covering the point ξ, whose diameter is less than ε.

Part a) does not differ from the case considered in (3), when the whole space has finite measure. Thus, part b) remains to be proved. This proof is based on the following lemma:

Lemma. If a set $M \subset R$ is the sum of an increasing sequence of measurable sets

$$M_1 \subset M_2 \subset \dots \subset M_n \subset \dots, \qquad M = \sum_{i=1}^{\infty} M_i,$$

then

$$\lim_{n\to\infty} \operatorname{mes}_\varepsilon M_n \geq \operatorname{mes}_\varepsilon M.$$

Proof. Two cases are possible:

$$1)\ \lim_{n\to\infty} \operatorname{mes}_\varepsilon M_n = \infty, \qquad 2)\ \lim_{n\to\infty} \operatorname{mes}_\varepsilon M_n < \infty.$$

In the first case the assertion of the lemma is obvious. Let us consider the second case. Fix $\delta > 0$. Cover each of the sets M_i by a system g_i of domains so that the sum of the elementary measures of these domains differs from the measure $\operatorname{mes}_\varepsilon M_i$ by less than $\delta/2^i$. From each covering g_i choose a finite system g_i^* so that the sum of the elementary measures of the domains from the system g_i^* differs from the sum of the elementary measures of the domains from the system g_i by less than $\delta/2^i$. Each covering g_i^* consists of a finite number n_i of domains

$$G_1^{i*}, \dots, G_{n_i}^{i*}.$$

Consider the covering g_1^*. There is a subsequence of the natural sequence $\alpha_{11} = 1, \alpha_{12}, \dots, \alpha_{1n}, \dots$ such that each domain of the system g_1^* either belongs to all the coverings $g_{\alpha_{1i}}^*$, or does not belong to any of them for $i > 1$. Consider the system

$g^*_{\alpha_{12}}$. For it there is likewise a sequence of numbers $\alpha_{21} = \alpha_{12}, \alpha_{22}, \alpha_{23}, ...$, which is a subsequence of the first sequence $\{\alpha_{1i}\}$, and each domain of the system $g^*_{\alpha_{12}}$ either belongs to each of the systems $g^*_{\alpha_{2i}}$ $(i > 1)$, or does not belong to any one of them. Analogous subsequences are found for each system $g_{\alpha_{i1}}$. As a result we obtain a sequence of sequences, each subsequent one of which is a subsequence of the preceding one.

Consider the diagonal sequence $\{\alpha_{i1}\}$, and henceforth we shall deal only with the sets and coverings from this sequence. Let $\{\Gamma^0_{i_k}\}$ be the system of domains from $\{\Gamma_i\}$, each of which belongs to all coverings $g^*_{\alpha_{i1}}$, beginning with some i. Now remove from each covering $g^*_{\alpha_{i1}}$ the domains that have entered the system $\{\Gamma^0_{i_k}\}$. Denote the remaining system of domains by $\tilde{g}_{\alpha_{i1}}$.

We shall now consider only the sets

$$\widetilde{M}_{\alpha_{i1}} = M_{\alpha_{i1}} \cap g_{\alpha_{i1}}$$

and their coverings $\tilde{g}_{\alpha_{i1}}$. Consider the set $\widetilde{M}_{\alpha_{11}}$ and its covering $\tilde{g}_{\alpha_{11}}$. Consider also the covering $\tilde{g}_{\alpha_{1+q,\,1}}$ for some q. The domains of the covering $\tilde{g}_{\alpha_{1+q,\,1}}$, with respect to the domains of the covering $\tilde{g}_{\alpha_{11}}$, are divided into three groups. In the first group $A^1_{\alpha_{1+q,\,1}}$ we include those domains of the covering $\tilde{g}_{\alpha_{1+q,\,1}}$ which are contained entirely inside the covering $\tilde{g}_{\alpha_{11}}$. In the second group $B^1_{\alpha_{1+q,\,1}}$ we include the domains from the covering $\tilde{g}_{\alpha_{1+q,\,1}}$ which contain both points of $\tilde{g}_{\alpha_{11}}$ and points of the complement of $\tilde{g}_{\alpha_{11}}$. And in the third group $C^1_{\alpha_{1+q,\,1}}$ we include the domains remaining in the system $\tilde{g}_{\alpha_{1+q,\,1}}$.

Consider the system of domains $A^1_{\alpha_{1+q,\,1}}$. It can be shown that there is such a sufficiently large number q_0 that the sum of the elementary measures of the domains $A^1_{\alpha_{1+q,\,1}}$

differs from the sum of the elementary measures of the system $\tilde{g}_{\alpha_{11}}$ by less than $(2\delta/2^{\alpha_{11}} + 2\delta/2^{\alpha_{1+q,1}})$.

Let us now consider the system of domains

$$\tilde{g}_{\alpha_{i_1},1} = \tilde{g}_{\alpha_{11}} + B^1_{\alpha_{1+q_0},1} + C^1_{\alpha_{1+q_0},1}.$$

We carry out with the system of domains $\tilde{g}_{\alpha_{i_1},1}$ the same operation that we have just carried out with the system $\tilde{g}_{\alpha_{11}} = \tilde{g}_{\alpha_{11}}$. In this way we obtain a system of domains $\tilde{g}_{\alpha_{i_2},1}$, and so on. Suppose we have a domain $\tilde{g}_{\alpha_{i_n},1}$. Construct the system of domains $\tilde{g}_{\alpha_{i_{n+1}},1}$. Consider the set $\widetilde{M}_{\alpha_{i_n},1}$ and its covering $\tilde{g}_{\alpha_{i_n},1}$. Consider also the covering $\tilde{g}_{\alpha_{i_n+q},1}$ for some q. The domains of the covering $\tilde{g}_{\alpha_{i_n+q},1}$, in relation to the domains of the covering $\tilde{g}_{\alpha_{i_n},1}$, are divided into three groups. In the first group $A^{i_n}_{\alpha_{i_n+q},1}$ we include those domains of the covering $\tilde{g}_{\alpha_{i_n+q},1}$ which are contained entirely inside the covering $\tilde{g}_{\alpha_{i_n},1}$. In the second group $B^{i_n}_{\alpha_{i_n+q},1}$ we include those domains from the covering $\tilde{g}_{\alpha_{i_n+q},1}$ which contain both points of $\tilde{g}_{\alpha_{i_n},1}$ and points of the complement of $\tilde{g}_{\alpha_{i_n},1}$. In the third group $C^{i_n}_{\alpha_{i_n+q},1}$ we include the remaining domains in the system $\tilde{g}_{\alpha_{i_n+q},1}$.

Consider the system of domains $A^{i_n}_{\alpha_{i_n+q},1}$. There is such a sufficiently large number q that the sum of the elementary measures of the domains $A^{i_n}_{\alpha_{i_n+q},1}$ differs from the sum of the elementary measures of the system $\tilde{g}_{\alpha_{i_n},1}$ by less than $(2\delta/2^{\alpha_{i_n},1} + 2\delta/2^{\alpha_{i_n+q},1})$.

Denote by

$$\tilde{g}_{\alpha_{i_{n+1}},1} = \tilde{g}_{\alpha_{i_n},1} + B^{i_n}_{\alpha_{i_n+q_0},1}.$$

We obtain a sequence of systems of domains

$$\left\{\tilde{g}_{\alpha_{i_1},1} \subset \tilde{g}_{\alpha_{i_2},1} \subset \cdots \subset \tilde{g}_{\alpha_{i_n},1} \subset \cdots\right\}.$$

Let us now consider

$$\sum_{j=1}^{\infty} \tilde{g}_{\alpha_{i_j},1} = T.$$

The sum of the elementary measures of the domains T differs from the limit of the sums of the elementary measures of the domains $\tilde{g}_{\alpha_i,1}$ by no more than 8δ.

At the beginning of the proof we discarded from the sets M_i those parts which did not enter the finite coverings g_i^*. Each time we thereby discarded parts of the sets M_i, the ε-measure of which did not exceed $\delta/2^i$.

Consider now

$$\sum_{i=1}^{\infty} M_i \setminus \widetilde{M}_i.$$

This set will have ε-measure not exceeding 2δ. Cover

$$\sum_{i=1}^{\infty} M_i \setminus \widetilde{M}_i$$

by a system of domains P, the sum of whose elementary measures does not exceed 2δ.

Let now $\{\Gamma_i'\} = S + T + P$. The system $\{\Gamma_i'\}$ will cover the set M. The sum of the elementary measures of the regions of this system does not exceed 12δ, and, since δ is arbitrary, the lemma is proved.

Proof of the theorem. Map our metric space R by a homeomorphic transformation into the Hilbert space K. Under this transformation our set M, which is an A-set in the metric space, passes into some A-set M^1 in the Hilbert space K (4). Define a measure of Hausdorff type in the Hilbert space by assigning to the images of the regions Γ_i the elementary measure of the Γ_i themselves. Then the measure of the set M^1 in the Hilbert space will be equal to the measure of the set M in the space R.

We shall prove the theorem for the set M^1 in the space K. Thereby the theorem will be proved for the sets M in the space R. Construct the topological product of the space K with a one-dimensional Euclidean space. Denote this product by L. Let our set M^1 be obtained as the projection into the space K of some G_δ situated in the space L. Let our G_δ, in turn, be the intersection of open sets G_i. Each open set G_i we can represent in the form of a sum of a countable number of increasing closed sets f_{ij}. Denote the projection of the set f_{ij} in the space R by F_{ij}; $\prod_i F_{ij}$ will be a closed set contained in the set M.

Now consider the intersection $F_{ij} \cap M^1$. Fix the numbers ε_1 and ε_2. According to the lemma, in each G_i we can choose such a closed set f_{ij_0} that

$$\left|\operatorname{mes}_{\varepsilon_1} F_{ij_0} \cap M - \operatorname{mes}_{\varepsilon_1} M\right| < \frac{\varepsilon_2}{2^i}.$$

Since $\prod_i (F_{ij_0} \cap M) = \prod_i F_{ij_0}$, it follows that $\prod_i (F_{ij_0} \cap M)$ is a closed set. Since for a closed set one can choose a finite subcover from any cover, it is true that if a closed set is the intersection of a sequence of sets, then the ε-measure of the intersection is equal to the limit of the ε-measures of the intersecting sets.

Fix N. Choose ε_1 so that $\operatorname{mes}_{\varepsilon_1} M > 2N$, and $\varepsilon_2 < N/2$. Then

$$\operatorname{mes}_{\varepsilon_1} \prod_i F_{ij_0} > \operatorname{mes}_{\varepsilon_1} \prod_i F_{ij} > N.$$

Thus the theorem is proved.

It still does not follow from this theorem that in every A-set of infinite measure one can choose a closed set of arbitrarily large finite measure. For the ordinary Hausdorff measure in n-dimensional Euclidean space this can be proved. Namely, the following theorem holds:

Theorem 2. Let M be an A-set of infinite Hausdorff measure μ. Then for every N there exists a closed set $F \subset M$ such that $\mu F = N$.

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References

[1] A. S. Besicovitch, Proc. Kon. Ned. Akad. v. Wet. Amsterdam, ser. A, **55**, No. 3, 339 (1952).
[2] R. L. Minlos, DAN, **81**, No. 5 (1951).
[3] M. E. Munroe, *Introduction to Measure and Integration*, Cambridge, 1953.
[4] F. Hausdorff, *Set Theory*, 1937, p. 220.

Note: Figure translations are in progress. See original paper for figures.

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