

**ON FUNCTIONALS IN
THE SPACES
 $\{(C_{s_1 \dots s_n})\}$
AND $\{(L^p_{s_1 \dots s_n})\}$, ANNIHILATING
ON GENERALIZED
POLYNOMIALS OF
MANY VARIABLES**

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Abstract

Full Text

MATHEMATICS

I. A. EZROKHI

**ON FUNCTIONALS IN THE SPACES $C_{s_1 \dots s_n}$
AND $L^p_{s_1 \dots s_n}$, ANNIHILATING ON GENERAL-
IZED POLYNOMIALS OF MANY VARIABLES**

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Let $V(f)$ be the remainder term of an approximation formula, linear ⁽¹⁾ on some Banach space E and exact on some finite-dimensional subspace Ω . Then $V(f)$ is linear on E and annihilates on Ω .

A number of works ⁽³⁻¹⁰⁾ are devoted to a convenient representation of $V(f)$, both on E and on its subspaces, in the case when Ω is the set of polynomials of degree not exceeding a certain degree.

In the present note, which generalizes the results of E. Ya. Remez ⁽³⁾ and of the author ⁽⁹⁾, results are given on the representation of $V(f)$ in the case when $E = C_{s_1 \dots s_n} (L^p_{s_1 \dots s_n})$ (for the notation and definitions see ⁽⁹⁾), and Ω is a certain collection of generalised polynomials.

Definition 1. Suppose n systems of linearly independent functions $\{u_{i,k_i}(x_i)\} \subset C_{2s_i-1}$, $k_i = 0, 1, \dots, s_i - 1$, $i = 1, \dots, n$, are chosen. Then by a **generalized polynomial with respect to x_i of rank not exceeding $s_i - 1$** we shall mean the function

$$\omega_{\mu_i}(x_1, \dots, x_n) = \sum_{k_i=0}^{\mu_i-1} c_{k_i}^i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) u_{i,k_i}(x_i) \quad (\mu_i \leq s_i),$$

where all $c_{k_i}^i$ are continuous.

Definition 2. By $W_{\mu_i} = W(u_{i,0}, \dots, u_{i,\mu_i-1})$ we shall mean the Wronskian determinant formed from the functions $u_{i,0}, \dots, u_{i,\mu_i-1}$, and we shall write its nonvanishing on $[a_i, b_i]$ as follows: $W_{\mu_i} \neq 0$.

Definition 3. We set

$$D_{\mu_i}(f) = D_{\mu_i, x_i}(f, x_i) = \frac{W(u_{i,0}(x_i), \dots, u_{i,\mu_i-1}(x_i), f(x_1, \dots, x_n))}{W(u_{i,0}(x_i), \dots, u_{i,\mu_i-1}(x_i))}.$$

Definition 4. Henceforth $V_i(f)$ is a functional annihilating on the collection $\{\omega_{\mu_i}\}$ ($\mu \leq s_i$), and $V(f)$ is a functional annihilating on the collection of generalized polynomials $\{\omega_{\mu_1 \dots \mu_n}\}$ of rank not exceeding $s_i - 1$ ($\mu_i \leq s_i$) with respect to x_i , $i = 1, \dots, n$.

Theorem 1. Suppose $V \in (C_{s_1 \dots s_n})^*$ (i.e., $V(f)$ is linear on $C_{s_1 \dots s_n}$), respectively $(L_{s_1 \dots s_n}^p)^*$. Then, if for every i ($i = 1, \dots, n$) $W_{s_i} \neq 0$, then

$$V(f) = \sum_{i=1}^n V_i(f),$$

where $V_i \in (C_{s_i})^*$, respectively $(L_{s_i}^p)^*$.

Theorem 2. Suppose $V_i \in (C_{s_i})^*$ and $W_{s_i} \neq 0$. Then

$$V_i(f) = \int_{K_n} \dots \int D_{s_i}(f) d^n g_{s_i}(x_1, \dots, x_n); \quad (1)$$

$$g_{s_i}(\bar{x}_1, \dots, \bar{x}_n) = \bar{V}_i(\gamma_{s_i, \bar{x}_1 \dots \bar{x}_n}) = \lim_{m \rightarrow \infty} V_i \left[\int_{b_i}^{x_i} \theta_{\bar{x}_i}^m(z) H_{s_i}(x_i, z) dz \cdot \prod_{j \neq i} \theta_{\bar{x}_j}^m(x_j) \right]^*,$$

where $H_{s_i}(x_i, z)$ is the Cauchy function ⁽²⁾ of the differential equation $D_{s_i}(f) = 0$,

$$\theta_{\bar{x}_i}^m(x_i) = \begin{cases} 1, & \text{if } x_i \leq \bar{x}_i, \quad a_i < \bar{x}_i, \\ 0, & \text{if } x_i \geq \bar{x}_i + \frac{1}{m} \text{ or } \bar{x}_i = a_i, \\ 1 - m(x_i - \bar{x}_i), & \text{if } \bar{x}_i < x_i < \bar{x}_i + \frac{1}{m} \quad (i = 1, \dots, n), \end{cases}$$

and the function $g_{s_i}(x_1, \dots, x_n)$ has the same properties as $g_i(x_1, \dots, x_n)$ in Theorem 1 ⁽⁹⁾.

Theorem 3. Let $V_i \in (L_{s_i}^p)^*$ and $W_{s_i} \neq 0$. Then

$$V_i(f) = \int \dots \int_{K_{n-1}} \int_{a_i}^{b_i} D_{s_i}(f) dx_{j \neq i}^{n-1} \beta_{s_i}(x_1, \dots, x_n) dx_i, \quad (2)$$

where **

$$\beta_{s_i}(\bar{x}_1, \dots, \bar{x}_n) = \frac{\partial}{\partial x_i} \bar{V}_i(\gamma_{s_i, \bar{x}_1 \dots \bar{x}_n})$$

and it has the same properties as $\beta_i(x_1, \dots, x_n)$ in Theorem 2 ⁽⁹⁾.

Theorem 4. Let $F \in (C_{\mu_i})^*$. Then

$$F(f) = \sum_{j_i=0}^{\mu_i-1} \int \dots \int_{K_{n-1}} \frac{\partial^{j_i} f(x_1, \dots, x_{i-1}, b_i, x_{i+1}, \dots, x_n)}{\partial x_i^{j_i}} \times \\ \times dx_{j \neq i}^{n-1} g_{j_i}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) + \int \dots \int_{K_n} \frac{\partial^{\mu_i} f}{\partial x_i^{\mu_i}} d^n g_{\mu_i}(x_1, \dots, x_n),$$

where

$$g_{j_i}(\dots, \bar{x}_{i-1}, \bar{x}_{i+1}, \dots) = \bar{F}(\psi_{j_i, \bar{x}_1, \dots, \bar{x}_{i-1}, b_i, \bar{x}_{i+1}, \dots, \bar{x}_n}) \quad (j_i = 0, \dots, \mu_i - 1),$$

$$g_{\mu_i}(\bar{x}_1, \dots, \bar{x}_n) = \bar{F}(\psi_{\mu_i, \bar{x}_1, \dots, \bar{x}_n})$$

and, for example ⁽⁹⁾,

$$\bar{F}(\psi_{\mu_i, \bar{x}_1, \dots, \bar{x}_n}) = \lim_{m \rightarrow \infty} F \left(\int_{b_i}^{x_i} \theta_{x_i}^m(z_i) \frac{(x_i - z_i)^{\mu_i-1}}{(\mu_i - 1)!} dz_i \prod_{j \neq i} \theta_{x_j}^m(x_j) \right),$$

with all functions g of bounded variation in all variables and continuous from the right inside K_n ^(7,8).

Let us now consider the differential complexes ^(3,7,8)

$$(f, \bar{f}_{x_i}^{(\mu_i)})_{\mu_i} = (f(x_1, \dots, x_n), f_{x_i}^{(j_i)}(x_1, \dots, x_{i-1}, b_i, x_{i+1}, \dots, x_n),$$

$$0 \leq j_i \leq \mu_i - 1, \bar{f}_{x_i}^{(\mu_i)}(x_1, \dots, x_n)),$$

* We do not dwell on the fact that the values of the function $g_{s_i}(x_1, \dots, x_n)$ may be changed on a countable set of points in each variable inside K_n , without thereby changing the functional. Henceforth this equality serves as the definition of g_{s_i} .

** Obviously, the values of the function β_{s_i} may be changed on a set of measure zero with respect to x_i , without thereby changing the functional (see, moreover, the preceding footnote). Henceforth this equality serves as the definition of β_{s_i} .

terms of which are connected by the relation:

$$f(x_1, \dots, x_n) = \sum_{j_i=0}^{\mu_i-1} f_{x_i}^{(j_i)}(\dots x_{i-1}, b_i, x_{i+1}, \dots) \frac{(x_i - b_i)^{j_i}}{j_i!} + \int_{b_i}^{x_i} \bar{f}_{x_i}^{(\mu_i)}(\dots x_{i-1}, z_i, x_{i+1}, \dots) \frac{(x_i - z_i)^{\mu_i-1}}{(\mu_i - 1)!} dz_i,$$

where the integral is understood in the sense of Lebesgue.

Let C''_{μ_i} be the space of complexes for which $\bar{f}_{x_i}^{(\mu_i)}$ are Borel-measurable (τ) , with norm equal to the greatest of the exact upper bounds of the absolute values of the members of the complex. Then we extend $F \in (C_{\mu_i})^*$ to C''_{μ_i} as follows:

$$F[(f, \bar{f}_{x_i}^{(\mu_i)})_{\mu_i}] = \sum_{j_i=0}^{\mu_i-1} \int_{K_{n-1}} \dots \int f_{x_i}^{(j_i)} d_{x_j \neq i}^{n-1} g_{j_i} + \int_{K_n} \dots \int \bar{f}_{x_i}^{(\mu_i)} d^n g_{\mu_i},$$

where the integrals from Theorem 4 are now understood in the Stieltjes-Lebesgue sense.

Theorem 5. Let $V_i \in (C_{s_i-k})^*$ ($1 \leq k \leq s_i$) and $W_{\mu_i} \neq 0$. Then for $f \in L^p_{\mu_i}$ ($s_i - k < \mu_i \leq s_i$), $V_i(f)$ has the form (2), and for $f \in C_{\mu_i}$ also the form (1), everywhere with replacement of the index s_i by μ_i . Moreover, for each m ($1 \leq m < \mu_i - s_i + k$) everywhere, and for $m = \mu_i - s_i + k$, apart from all possible exclusions with respect to x_i ,

$$\frac{\partial^{m-1} \beta_{\mu_i}(\bar{x}_1, \dots, \bar{x}_n)}{\partial \bar{x}_i^{m-1}} = -V_i[(\lambda_{\mu_i-m, \bar{x}_1 \dots \bar{x}_n}^{(\mu_i)}, \lambda_{0, \bar{x}_1 \dots \bar{x}_n}^{(\mu_i)})_{\mu_i-m}],$$

where the right-hand side is equal to zero for $\bar{x}_i = a_i$, $\bar{x}_i = b_i$ and is continuous on the right inside K_n , while

$$g_{\mu_i} = \int_{a_i}^{x_i} \beta_{\mu_i} dx_i$$

and

$$\lambda_{\mu_i-m, \bar{x}_1 \dots \bar{x}_n}^{(\mu_i)}(x_1, \dots, x_n) = \prod_{j=1}^n \theta_{\bar{x}_j}(x_j) \frac{\partial^{m-1} H_{\mu_i}(x_i, \bar{x}_i)}{\partial \bar{x}_i^{m-1}},$$

$$\lambda_{0, \bar{x}_1 \dots \bar{x}_n}^{(\mu_i)}(x_1, \dots, x_n) = \prod_{j=1}^n \theta_{\bar{x}_j}(x_j) \frac{\partial^{\mu_i-1} H_{\mu_i}(x_i, \bar{x}_i)}{\partial x_i^{\mu_i-m} \partial \bar{x}_i^{m-1}}.$$

Theorem 6. If $V_i \in (C_{s_i-\mu_i})^*$, $s_i \geq \mu_i > k \geq 1$, $W_{s_i} \neq 0$, $W_{s_i-k} \neq 0$, then

$$\beta_{s_i}(\bar{x}_1, \dots, \bar{x}_n) = - \int_{a_i}^{x_i} H_{s_i, s_i-k}(z_i, \bar{x}_i) \beta_{s_i-k}(\bar{x}_1, \dots, z_i, \dots, \bar{x}_n) dz_i,$$

where $H_{s_i, s_i-k}(x_i, \bar{x}_i)$ is the Cauchy function of the differential operator Q_{s_i, s_i-k} ,

$$D_{s_i}(f) = Q_{s_i, s_i-k}(D_{s_i-k}(f)).$$

The last two theorems also make it possible to judge the number of sign changes of the functions $\beta'_{\mu_i} ({}^4, {}^8, {}^9)$.

It remains to note that, under the condition $u_{i,0} = 1$ ($i = 1, \dots, n$), there hold assertions analogous both in form and in proof

all the results established in ⁽⁹⁾ (pp. 399-413), with the corresponding replacement of words and symbols: “polynomial,” “degree,” $\varphi_{n-1, \bar{x}}(x)$, $\partial^\mu f / \partial x^\mu$ by “generalized polynomial,” “rank,” $\lambda_{\mu-1, \bar{x}}^{(\mu)}(x)$, $D_\mu(f)$ (if $W_\mu \neq 0$).

Example. Suppose a cubature formula is given

$$\frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} f(x_1, x_2) dx_1 dx_2 - \sum_{k=1}^9 A_k f(x_1^k, x_2^k) = V(f),$$

$$A_1 = A_2 = A_3 = A_4 = \left(\frac{1}{2} - \frac{1}{\pi}\right)^2,$$

$$A_5 = A_6 = A_7 = A_8 = \frac{2\pi - 4}{\pi^2}, \quad A_9 = \frac{12 - 4\pi}{\pi^2},$$

$$-x_1^1 = x_1^2 = x_1^3 = -x_1^4 = -x_2^1 = -x_2^2 = x_2^3 = x_2^4 = \frac{\pi}{2},$$

$$-x_1^5 = x_1^7 = -x_2^6 = x_2^8 = \frac{\pi}{3},$$

$$x_1^6 = x_1^8 = x_1^9 = x_2^5 = x_2^7 = x_2^9 = 0.$$

It is exact for generalized polynomials of rank not higher than 3 in each variable, where for $i = 1, 2$

$$u_{i,0}(x_i) \equiv 1, \quad u_{i,1}(x_i) = x_i, \quad u_{i,2}(x_i) = \cos x_i, \quad u_{i,3}(x_i) = \sin x_i.$$

Then for $f \in C_{4,4}$

$$V(f) = \frac{1}{144}(105\pi^2 - 520\pi + 644) \left\{ \left(\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial x^2} \right) f(\xi_1, \eta_1) + \left(\frac{\partial^4}{\partial y^4} + \frac{\partial^2}{\partial y^2} \right) f(\xi_2, \eta_2) \right\} \\ - \frac{1}{144}(117\pi^2 - 560\pi + 652) \left\{ \left(\frac{\partial^4}{\partial x^4} + \frac{\partial^2}{\partial x^2} \right) f(\xi_3, \eta_3) + \left(\frac{\partial^4}{\partial y^4} + \frac{\partial^2}{\partial y^2} \right) f(\xi_4, \eta_4) \right\}.$$

Ukrainian Agricultural Academy

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Note: Figure translations are in progress. See original paper for figures.

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