ON THE FIRST BOUNDARY-VALUE PROBLEM OF THE NONLINEAR THEORY OF ELASTICITY

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ON THE FIRST BOUNDARY-VALUE PROBLEM OF THE NONLINEAR THEORY OF ELASTICITY

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In the present note we consider existence and uniqueness theorems for the nonlinear problem of the theory of elasticity in the case when the displacement vector is prescribed on the boundary of the body.

We consider the three-dimensional problem of the nonlinear theory of elasticity for the pair of spaces $(W_2^4; W_2^2)$, where W_l^k is the space of functions having k-th generalized derivatives summable with power l, and the plane problem of the nonlinear theory of elasticity for the pair of spaces $(E_{\lambda',2}; E_\lambda)$, where $E_{\alpha,k}$ is the space of functions whose k-derivatives satisfy the Lipschitz condition with exponent α. In what follows we set $E_\lambda = E_{\lambda,0}$.

We introduce the norm in the space W_2^4 as follows: if $f = (f_1, f_2, f_3)$, then we put

$$\|f\|^2_{W_2^4} = \sum_{i=1}^{3} \left\{ \sum_{k_1+k_2+k_3=4} \left\| \frac{\partial^4 f_i}{\partial x_1^{k_1} \partial x_2^{k_2} \partial x_3^{k_3}} \right\|_{L_2} + \sum_{l_1+l_2+l_3=3} \left\| \frac{\partial^3 f_i}{\partial x_1^{l_1} \partial x_2^{l_2} \partial x_3^{l_3}} \right\|_{L_2} + \sum_{\alpha_1+\alpha_2+\alpha_3=2} \left\| \frac{\partial^2 f_i}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} \right\|_{C} + \right.$$

The regularity of introducing such a norm follows from the embedding theorems of S. L. Sobolev (1).

Let P be the operator of the nonlinear theory of elasticity; $\mathbf{Pu} = (P_{xu}, P_{yu}, P_{zu})$, then

$$\mathbf{Pu} = \operatorname{div} T + \mathbf{K}; \qquad T = \psi(\Gamma^2)E + \left(k - \frac{1}{3}\psi(\Gamma^2)\right)\varepsilon I,$$

where $T = (\tau_{ik})$ is the stress tensor; $E = (\varepsilon_{ik})$ is the strain tensor; I is the unit tensor; $\mathbf{K} = (X, Y, Z)$ is the vector of body forces; $\mathbf{u}$ is the displacement vector; $\psi(\Gamma^2)$ is the so-called plasticity function; Γ is the strain intensity; $\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z$ is the first invariant of the strain tensor.

Let $P'_{(u_0)}$ be the Fréchet differential operator for the operator P. Consider the problem

$$P\mathbf{u} = 0; \qquad \mathbf{u}|_S = 0, \tag{1}$$

where S is the boundary of the region of the elastic body.

If Newton' s method (^{2}) is applied to this problem, then successive approximations to the solution of our problem may be obtained from the following system:

$$P'_{(u_n)}\mathbf{u}_{n+1} = P'_{(u_n)}\mathbf{u}_n - P\mathbf{u}_n. \tag{*}$$

If, in particular, we put $\psi(\Gamma^2) = 2p(1 - \omega(\Gamma^2))$, where $\omega(\Gamma^2)$ is a certain new function, and write the system for the modified Newton method (^{2}), taking $\mathbf{u} = 0$ as the zeroth approximation, then instead of $(*)$ we obtain the system

$$P'_{(0)}\mathbf{u}_{n+1} = P'_{(0)}\mathbf{u}_n - P\mathbf{u}_n.$$

This system coincides with the system obtained by the method of elastic solutions (3), which, in this way, may be regarded as a special case of the modified Newton method.

The principal results obtained by us for the pair of spaces (W_2^4, W_2^2) may be formulated in the form of the following propositions.

Lemma 1. *If the function $\psi(\Gamma^2)$ is such that the inequalities*

$$\psi(\Gamma^2) > 0; \tag{1}$$

$$\psi'(\Gamma^2) < 0; \tag{2}$$

$$\psi(\Gamma^2) > -2\psi'(\Gamma^2)(\gamma_{xy}^2 + \gamma_{xz}^2 + \gamma_{yz}^2); \tag{3}$$

$$\begin{aligned} \psi(\Gamma^2) > -\frac{8}{9}\psi'(\Gamma^2)\big[&(\varepsilon_x - \varepsilon_y)^2 + (\varepsilon_x - \varepsilon_z)^2 + (\varepsilon_y - \varepsilon_z)^2 \\ &+ (\varepsilon_x - \varepsilon_y)(\varepsilon_x - \varepsilon_z) + (\varepsilon_y - \varepsilon_x)(\varepsilon_y - \varepsilon_z) \\ &+ (\varepsilon_z - \varepsilon_x)(\varepsilon_z - \varepsilon_y)\big], \end{aligned} \tag{4}$$

are satisfied, then the operator $-P'_{(u_0)}$ is positive definite in L_2.

Let us note that conditions (3) and (4) will be all the more satisfied if one requires that the condition formulated by A. A. Ilyushin (3) be satisfied: $dT/d\Gamma > 0$, where T is the stress intensity.

Lemma 2. *Let the boundary S of the domain Ω, occupied by the elastic body, have 8 continuous derivatives. Then the equation $P'_{(0)}\mathbf{u} = \mathbf{f}$, where $\mathbf{f} \in W_2^2$, is uniquely solvable, and for the solution of this equation $\mathbf{u}$ the inequality*

$$\|\mathbf{u}\|_{W_2^4(\Omega)} \leq B_0 \|\mathbf{f}\|_{W_2^2(\Omega)}.$$

is satisfied.

In proving this lemma, work (4) is used.

We now introduce into the plasticity function $\psi(\Gamma^2)$ a small parameter $\varkappa$, putting $\psi(\Gamma^2) = 2\rho(1 - \varkappa\omega(\Gamma^2))$.

Lemma 3. *In any sphere of the space W_2^4, $\|\tilde{\mathbf{u}}\|_{W_2^4(\Omega)} \leq \text{const}$, the second Fréchet differential $P''_{(\tilde{\mathbf{u}})}$ is bounded in norm by a constant depending, generally speaking, on $\varkappa$, namely:*

$$\|P''_{(\tilde{\mathbf{u}})}\mathbf{u}\mathbf{u}'\|_{W_2^2(\Omega)} \leq N\varkappa \|\mathbf{u}\|_{W_2^4(\Omega)} \|\mathbf{u}'\|_{W_2^4(\Omega)};$$

N is a constant depending on $\|\tilde{\mathbf{u}}\|_{W_2^4}$.

Theorem 1. *If the boundary S of the domain Ω is continuously differentiable 8 times and if $\mathbf{K} = (X, Y, Z) \in W_2^2(\Omega)$, then for $\varkappa \leq (2B_0^2 N\eta_0)^{-1}$ there exists a unique solution of problem (I), $\mathbf{u}^*$, which lies in the neighborhood of 0 determined by the inequality*

$$\|\mathbf{u}\|_{W_2^4(\Omega)} \leq \frac{1 - \sqrt{1 - 2h}}{h} B_0\eta_0, \qquad B_0 = \text{const}.$$

For $\varkappa < (2B_0^2 N\eta_0)^{-1}$, the modified Newton process converges to the solution with a rate determined by the inequality

$$\|\mathbf{u}'_n - \mathbf{u}^*\| \leq q^{n-1}\|\mathbf{u}_1 - \mathbf{u}^*\|_{W_2^4}, \qquad \text{where } q = 1 - \sqrt{1 - 2h} < 1.$$

Here $\eta_0 = \|\mathbf{K}\|_{W_2^2(\Omega)}$; $h = B_0^2\eta_0 N\varkappa$; $\mathbf{u}'_n$ is the approximate solution corresponding to the modified Newton process; $\mathbf{u}_1$ is the first approximation in the basic Newton process.

Let us now turn to the pair of spaces $(E_{\lambda,2}; E_\lambda)$ and consider the plane problem of nonlinear elasticity.

We introduce the norm in the space $E_{\lambda,m}$ in the usual way, setting

$$\|f\|_{\lambda,m} = \sum_{i+k=0}^{m} \frac{1}{i!k!} \left\| \frac{\partial^{i+k} f}{\partial x_1^i \partial x_2^k} \right\|_{\lambda}, \qquad \|f\|_{\lambda} = \max_{\overline{\Omega}} |f| + \min\{C_f\},$$

where the constants λ and C_f are determined from the inequality

$$|f(M_1) - f(M_2)| \leq C_f |M_1 - M_2|^{\lambda};$$

M_1, M_2 are points of the domain of definition of the function f. If $f = (f_1, f_2)$, then we put

$$\|f\|_{\lambda,m} = \sum_{k=1}^{2} \|f_k\|_{\lambda,m}.$$

The operator of the plane nonlinear theory of elasticity that interests us here, $\widetilde{P}$, will be obtained from the operator P already considered, if we set $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = u_z = 0$.

Consider the problem

$$\widetilde{P}\mathbf{u} = 0, \qquad \mathbf{u}|_S = 0. \tag{I'}$$

The principal results obtained for the pair of spaces $(E_{\lambda,2}; E_{\lambda})$ are given by the following propositions.

Lemma 4. In any sphere of the space $E_{\lambda,2}$, $\|\mathbf{u}\|_{\lambda,2} \leq \text{const}$, the second Fréchet differential $\widetilde{P}''_{(\mathbf{u})}$ is bounded in norm by a constant which, in general, depends on x, namely

$$\left\| \widetilde{P}''_{(\mathbf{u})} \mathbf{u}\mathbf{u}' \right\|_{\lambda} \leq N_x \|\mathbf{u}'\|_{\lambda,2} \|\mathbf{u}''\|_{\lambda,2}.$$

Lemma 5. Let the domain occupied by the elastic body be the disk of unit radius. Then the operator $\widetilde{P}'_{(0)}$ has an inverse $[\widetilde{P}'_{(0)}]^{-1}$, and

$$\left\| [\widetilde{P}'_{(0)}]^{-1} \right\|_{\lambda,2} \leq B_0.$$

In the proof of this lemma the ideas of work (^{5}) are used.

Theorem 2. If the domain occupied by the body is the disk of unit radius and if $\mathbf{K} = (X, Y) \in E_{\lambda}$, then for $x \leq (2B_0^2 N \eta_0)^{-1}$ there exists a unique solution of problem (I'), $\mathbf{u}^*$, lying in a neighborhood of 0 determined by the inequality

$$\|\mathbf{u}\|_{\lambda,2} \leq \frac{1 - \sqrt{1 - 2h}}{h} B_0 \eta_0.$$

For $x < (2B_0^2 N\eta_0)^{-1}$, the modified Newton process converges to the solution $\mathbf{u}^*$ with the rate determined by the inequality

$$\|\mathbf{u}'_n - \mathbf{u}^*\| \leq q^{n-1}\|\mathbf{u}_1 - \mathbf{u}^*\|_{\lambda,2}, \qquad \text{where } q = 1 - \sqrt{1-2h} < 1.$$

Here $\eta_0 = \|\mathbf{K}\|_\lambda,\ h = B_0^2\eta_0 N x$.

Let us now consider an arbitrary plane domain, whose boundary L we require to be such that the angle between the Ox axis and the radius vector drawn through the boundary point (x_1, y_1), and the angle between this radius vector and the tangent to the contour at the point (x_1, y_1), belong to the space E_λ, while the angle between the normal at the point (x_1, y_1) and the Ox axis belongs to the space $E_{\lambda,1}$. Then the following propositions hold.

Lemma 6. Let $\mathbf{u}^{(1)} = (u_x^{(1)}, u_y^{(1)})$ be the vector whose components are the regular components of the Green tensor for the problem of the linear theory of elasticity. Then the estimates

$$|D_1\mathbf{u}^{(1)}(x,y;\xi,\eta)| \leq \frac{C_1}{r_{02}}, \qquad |D_2\mathbf{u}^{(1)}(x,y;\xi,\eta)| \leq \frac{C_2}{r_{02}^2},$$

$$|D_3\mathbf{u}^{(1)}(x,y;\xi,\eta)| \leq \frac{C_3}{r_{02}^3},$$

where $r_{02}^2 = (x-\xi)^2 + (y-\eta)^2$; C_k are constants; D_k is the k-th derivative of $\mathbf{u}^{(1)}$.

Lemma 7. If the boundary L of the domain satisfies the conditions formulated above, then the operator $\widetilde{P}'_{(0)}$ has an inverse, acting from E_λ to $E_{\lambda',2}$, and such that $\|[\widetilde{P}'_{(0)}]^{-1}\|_{\lambda',2} \leq B_0,\ \lambda' < \lambda$.

Theorem 3. If the vector $\mathbf{K} \in E_\lambda$ and the boundary L of the domain is of the type indicated above, then for $\varkappa \leq (2B_0^2 N\eta_0)^{-1}$ there exists a unique solution $\mathbf{u}^*$ of problem (I'), lying in a neighborhood of 0 determined by the inequality

$$\|\mathbf{u}\|_{\lambda',2} \leq \frac{1-\sqrt{1-2h}}{h}\, B_0\eta_0.$$

For $\varkappa < (2B_0^2 N\eta_0)^{-1}$ the modified Newton process converges to the solution with the rate determined by the inequality

$$\|\mathbf{u}'_n - \mathbf{u}^*\|_{\lambda',2} \leq q^{n-1}\|\mathbf{u}_1 - \mathbf{u}^*\|_{\lambda',2},$$

where

$$q = 1 - \sqrt{1 - 2h} < 1.$$

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Note: Figure translations are in progress. See original paper for figures.

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