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Abstract

Full Text

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INVERSE THEOREMS OF THE CONSTRUCTIVE THEORY OF FUNCTIONS DEFINED ON A FINITE INTERVAL OF THE REAL AXIS

(Presented by Academician S. N. Bernstein on 27 IV 1957)

It is known ([1,5]) that the direct theorems of the constructive theory of functions of a real variable, established by Jackson and subsequently supplemented by Zygmund ([4]), in the case of a finite interval do not admit a complete converse. The reason for this is explained by the following general theorem ([6]) (for a special case see ([8])), established by us in 1950 and showing that the aforementioned result of Jackson, while order-sharp from the point of view of approximations on the whole real axis ([1,7]), becomes crude when one passes to approximations by algebraic polynomials on a finite interval.

Theorem 1. If a function $f(x)$, defined on $[-1,1]$, has a continuous derivative of order r ($r \geq 0$), then there exists a constant C, independent of x and n, such that for every $n = 1, 2, ...$ there is an ordinary polynomial $P_n(x)$ of degree not exceeding n, satisfying, for each $x \in [-1,1]$, the inequality

$$|f(x) - P_n(x)| \leq \frac{c}{n^r}\left(\sqrt{1-x^2} + \frac{|x|}{n}\right)^r \omega_r\left[\frac{1}{n}\sqrt{1-x^2} + \frac{|x|}{n}\right], \qquad (1)$$

where

$$\omega_r(h) = \omega(f^r; h) = \sup_{|x_1 - x_2| < h} |f^{(r)}(x_1) - f^{(r)}(x_2)|, \qquad x_1,\ x_2 \in [-1,1],$$

is the modulus of continuity of the r-th derivative.

This assertion shows that, while preserving the best order of approximation over the interval as a whole, near its endpoints this order can be substantially improved. Moreover, the character of the improvement of the approximations near the endpoints of the interval is here the same as the character of the deterioration of the order of maximal growth of the derivatives of the algebraic polynomial $P_n(x)$ as $|x| \to 1$ in the well-known inequalities of S. N. Bernstein and A. A. Markov (see, for example, ([1])). These inequalities were later substantially supplemented by S. N. Bernstein ([2]), who obtained the following result.

Theorem 2 (S. N. Bernstein). If $t(x) > 0$ is a continuous function, then from the inequality $|P_n(x)| \leq t(x)$ on the interval $[-1,1]$ there follows the inequality

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$$|P_n'(x)| \leq Cn \min\left\{\frac{t(x)}{\sqrt{1-x^2}},\ n\max[t(+1),\, t(-1)]\right\}, \tag{2}$$

where C is a constant independent of x and n.*

* In this theorem of S. N. Bernstein, as $n \to \infty$, the constant C may be replaced by the quantity $1+\varepsilon_n \to 1$ ([2]).

Recently Yu. A. Brudnyi gave a simple proof of inequality (2), also applicable to the case when $t(x) = t_n(x)$ is the right-hand side of (1). Using this inequality, one can obtain the following theorem, inverse to Theorem 1 (in the case $r = 0$).

Theorem 3. *If for a function $f(x)$, defined on $[-1,1]$, with a certain modulus of continuity $\omega(h)$, one can indicate a sequence of ordinary polynomials $P_n(x)$ $(n = 1, 2, \ldots)$ such that*

$$|f(x) - P_n(x)| \leq \omega\left[\frac{1}{n}\left(\sqrt{1-x^2} + \frac{|x|}{n}\right)\right], \qquad x \in [-1,1], \tag{3}$$

then there exists a positive constant C, independent of h, such that

$$\omega(f,h) \leq Ch\int_h^1 \frac{\omega(u)}{u^2}\,du, \qquad 0 < h \leq \frac{1}{2}. \tag{4}$$

An inequality analogous to (4), for best approximations by trigonometric polynomials of periodic functions (with respect to the metric L_p), in a somewhat more general form was given earlier in the work of M. F. Timan and the author ([11]), Theorem 4*.

In proving Theorem 3, the well-known method of S. N. Bernstein for obtaining inverse theorems is essentially followed. Let $0 \leq \delta \leq 1/n$. It is obvious that if $0 \leq x \leq x+\delta \leq 1$, then for any integer $m > 0$

$$|\Delta_\delta f(x) - \Delta_\delta P_{2^{m+1}}(x)| \leq 2\omega\left[\frac{1}{2^{m+1}}\left(\sqrt{1-x^2} + \frac{|x|}{2^{n+1}}\right)\right].$$

It follows from this that for any $x \in [-1,1]$ and $n \geq 2$ there is a natural number

$$m = m(x,n) = \begin{cases} [\ln n/2\ln 2], & \text{if } |x| \geq 1 - \dfrac{2}{n}, \\ [\ln(n\sqrt{1-x^2})/\ln 2], & \text{if } |x| < 1 - \dfrac{2}{n}, \end{cases}$$

such that

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$$|\Delta_\delta f(x) - \Delta_\delta P_{2^{m+1}}(x)| \leq 4\omega\left(\frac{2}{n}\right). \tag{5}$$

Furthermore, for any $x \in [-1, 1]$,

$$|P'_{2^{m+1}}(x)| \leq |P'_2(x)| + \sum_{k=1}^{m} |P'_{2^{k+1}}(x) - P'_{2^k}(x)|,$$

$$|P_{2^{k+1}}(x) - P_{2^k}(x)| \leq 2\omega\left[\frac{1}{2^k}\left(\sqrt{1-x^2} + \frac{|x|}{2^k}\right)\right]. \tag{6}$$

Applying Theorem 2 to the function

$$t_{2^k}(x) = \omega\left(\frac{\sqrt{1-x^2}}{2^k}\right) + \omega(2^{-2k}),$$

we obtain that there exists a constant $C_1 > 0$ such that

$$|P'_{2^{k+1}}(x) - P'_{2^k}(x)| \leq$$

$$\leq C_1 \min\begin{cases} 2^k\sqrt{(1-x^2)^{-1}}\,\omega(2^{-k}\sqrt{1-x^2}) + 2^k\sqrt{(1-x^2)^{-1}}\,\omega(2^{-2k}), \\ 2^{2k}\omega(2^{-2k}). \end{cases} \tag{7}$$

We assume $x \geq 0$. Since for $x \geq 1 - 2/n$ one has $(1-x^2)^{-1/2} \geq 2^{m-1}$, there exists a positive constant C_2 having the property that, for all such values of x and for $1 \leq k \leq m$, the inequalities

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* The indicated inequality subsequently played a well-known role in a number of papers by S. B. Stechkin (see ([12]), Theorem 8 and the propositions adjoining it, as well as some of his papers on the absolute convergence of orthogonal series).

$$|P'_{2^{k+1}}(x) - P'_{2^k}(x)| \leq C_2 \cdot 2^{2k}\omega\left(2^{-2k}\right)$$

and, by virtue of (6),

$$|P'_{2^{m+1}}(x)| \leq C_3 \sum_{k=1}^{m} 2^{2k}\omega\left(2^{-2k}\right) \leq C_4 \sum_{k=1}^{2^{2m}} \omega\left(\frac{1}{k}\right) \leq C_4 \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right). \tag{8}$$

Let now $x < 1-2/n$, $q = [-\ln(1-x^2)/2\ln 2]$. From inequality (7) there follows the existence of a positive constant C_5 such that, for all $k \le q$, the inequalities

$$|P'_{2^{k+1}}(x) - P'_{2^k}(x)| \le C_5 2^{2k} \omega\left(2^{-2k}\right),$$

and of a constant C_6 such that, for all $k > q$,

$$|P'_{2^{k+1}}(x) - P'_{2^k}(x)| \le C_6 \cdot 2^k \sqrt{(1-x^2)^{-1}}\, \omega\left(2^{-k}\sqrt{1-x^2}\right).$$

Consequently, from (6), for $x < 1-2/n$ we obtain

$$|P'_{2^{m+1}}(x)| \le C_7 \sum_{k=1}^{q} 2^{2k} \omega\left(2^{-2k}\right) + C_6 \sqrt{(1-x^2)^{-1}} \sum_{k=q+1}^{m} 2^k \omega\left(2^{-k}\sqrt{1-x^2}\right) \le$$

$$\le C_8 \sum_{k=1}^{2^{2q}} \omega\left(\frac{1}{k}\right) + C_9 \sum_{k=q}^{m} 2^{k+q} \omega\left(2^{-k-q}\right) \le C_8 \sum_{k=1}^{2^{2q}} \omega\left(\frac{1}{k}\right) + C_{10} \sum_{k=1}^{2^{m+q}} \omega\left(\frac{1}{k}\right),$$

or, since for all values of x considered here $2^{m+q} \le n$,

$$|P'_{2^{m+1}}(x)| \le C_{11} \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right).$$

Thus, from (5) we obtain the estimate

$$|f(x) - f(x+\delta)| = |\Delta_\delta f(x)| \le \frac{C_0}{n} \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right) + 4\omega\left(\frac{2}{n}\right).$$

It remains to take into account that

$$\omega\left(\frac{2}{n}\right) \le \frac{4}{n} \sum_{k=1}^{[n/2]} \omega\left(\frac{1}{k}\right) \le \frac{4}{n} \sum_{k=1}^{n} \omega\left(\frac{1}{k}\right).$$

Corollary 1. If $f(x)$ satisfies the conditions of Theorem 3 and

$$h \int_h^1 \frac{\omega(u)}{u^2}\, du = O[\omega(h)],$$

then $\omega(f; h) = O[\omega(h)]$.

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In an analogous way, using inequality (2), one can obtain the following theorem, inverse to Theorem 1 (for $r \geq 1$).

Theorem 4. If, for a function $f(x)$ defined on $[-1,1]$, with some modulus of continuity $\omega(h)$ satisfying the condition

$$\int_0^1 \frac{\omega(u)}{u}\, du < \infty,$$

one can specify a sequence of ordinary polynomials $P_n(x)$ $(n = 1, 2, \ldots)$ such that

$$|f(x) - P_n(x)| \leq \frac{1}{n^r}\left(\sqrt{1-x^2} + \frac{|x|}{n}\right)^r \omega\left[\frac{1}{n}\left(\sqrt{1-x^2} + \frac{|x|}{n}\right)\right], \quad x \in [-1,1],$$

then $f(x)$ has on the given interval an r-th continuous derivative $f^{(r)}(x)$ and

$$\omega(f^{(r)}; h) \leq C\left\{h\int_h^1 \frac{\omega(u)}{u^2}\, du + \int_0^h \frac{\omega(u)}{u}\, du\right\}, \quad 0 < h \leq \tfrac{1}{2}. \tag{9}$$

An inequality analogous to inequality (9), for best approximations of periodic functions by trigonometric polynomials, was obtained by us earlier in a more general form in [10] (see also [12], Theorem 11). In the same direction (for moduli of continuity of higher orders) inequalities (9) and (4) can be generalized.

Corollary 2. If $f(x)$ satisfies the conditions of Theorem 4 and

$$h\int_h^1 \frac{\omega(u)}{u^2}\, du = O[\omega(h)], \qquad \int_0^h \frac{\omega(u)}{u}\, du = O[\omega(h)], \tag{10}$$

then $f(x)$ has a derivative $f^{(r)}(x)$, continuous on $[-1,1]$, for which $\omega(f^{(r)}, h) = O[\omega(h)]$.

Consideration of the particular cases $\omega(u) = u^\alpha$ $(0 < \alpha < 1)$ and $\omega(u) = u^{r+\alpha}$ $(r \geq 1,\ 0 < \alpha < 1)$, respectively, in Corollaries 1 and 2 shows that, for functions having an r-th $(r \geq 0)$ derivative satisfying a Lipschitz condition of order α $(0 < \alpha < 1)$, the strengthened Jackson theorem (Theorem 1), with respect to order, is final and in this form admits a complete converse*. The same assertion remains valid in the more general case.

Theorem 5. Let $\omega(h)$ satisfy conditions (10) (respectively, the first of these conditions). A function $f(x)$, given on $[-1,1]$, has there an r-th derivative with modulus of continuity of order $O[\omega(h)]$ (respectively, a modulus of continuity

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of order $O[\omega(h)]$) if and only if there exists a sequence of polynomials $P_n(x)$ satisfying inequality (1), where $\omega_r(h) = \omega(h)$ (respectively, for $r = 0$).

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* The indicated particular case of Corollaries 1 and 2 in connection with Theorem 1 was recently considered by V. K. Dzyadyk [13].

Note: Figure translations are in progress. See original paper for figures.

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