ON FUNCTIONS WITHOUT COMMON VALUES AND THE OUTER BOUNDARY OF THE RANGE OF VALUES OF A FUNCTION

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Abstract

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MATHEMATICS

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ON FUNCTIONS WITHOUT COMMON VALUES AND THE OUTER BOUNDARY OF THE RANGE OF VALUES OF A FUNCTION

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In the author' s papers ([1,2]) a number of theorems on univalent conformal mappings were obtained. Here analogous questions are solved for non-univalent mappings.

In what follows B is a finitely connected domain of the z-plane with nondegenerate boundary continua and not containing the point $z=\infty$. By meromorphic, in particular regular in the domain B, functions we mean single-valued meromorphic or regular functions in B. $R_a(B)$ is the family of all functions $f(z)$, regular in the domain B and satisfying in it the conditions $|f(z)|<1,\ f(a)=0$, where a is a prescribed point of B. By $F(z,a)$ we denote that function of the class $R_a(B)$ for which $|f'(a)| \leq F'(a,a),\ f(z) \in R_a(B)$. The functions $f_\nu(z)$, $\nu=1,\dots,n$, defined respectively in domains B_ν (arbitrary ones), will be called **functions without common values** in these domains if $f_\nu(z_\nu) \neq f_\mu(z_\mu)$ for arbitrary $z_\nu \in B_\nu,\ z_\mu \in B_\mu$ and arbitrary $\nu \neq \mu;\ \nu,\mu = 1,\dots,n$.

Theorem 1. *If $f_\nu(z)$, $\nu=1,2$, in the domain B are meromorphic functions without common values, then for any points $z_\nu \in B$ at which $f_\nu(z)$ are regular, we have the sharp estimate*

$$|f_1'(z_1)f_2'(z_2)| \leq |f_1(z_1)-f_2(z_2)|^2\,F'(z_1,z_1)F'(z_2,z_2).$$

Extremal systems of functions are determined by the equations

$$(f_1(z)-a_1)/(f_1(z)-a_2) = \rho\varepsilon_1 F(z,z_1), \qquad (f_2(z)-a_2)/(f_2(z)-a_1) = \varepsilon_2 F(z,z_2)/\rho$$

for arbitrary finite and distinct a_1 and a_2, $\rho>0,\ |\varepsilon_1|=|\varepsilon_2|=1$.

Let any finite numbers a_1 and a_2, $a_1 \neq a_2$, be given, and let $\mathfrak{M}(a_1,a_2;a,B)$ be the class of all systems of functions $f_\nu(z)$, $\nu=1,2$, $f_\nu(a)=a_\nu$, $a \in B$, meromorphic and without common values in B. Denote by $\mathfrak{E}(a_1,a_2;a,B)$ the set of all points $M(X,Y) \equiv M(|f_1'(a)|,\ |f_2'(a)|)$ of the plane XOY in the class $\mathfrak{M}(a_1,a_2;a,B)$.

Theorem 2. *The set $\mathfrak{E}(a_1, a_2; a, B)$ is a closed domain with the excluded boundary point ∞*

$$0 \leq XY \leq |a_1 - a_2|^2 {F'}^2(a, a).$$

The set of all values w assumed in the domain B by a function $w = f(z)$ meromorphic in it and considered in the w-plane will be called the **range of values** of $f(z)$ in B.

In the case of an arbitrary number of functions, some known theorems on univalent mappings onto mutually nonoverlapping domains can, analogously to the preceding, be carried over to the case of functions without common values in the domain B, under the condition that the system of the ranges of values of these functions has a filling-in ([1]).

For any given distinct points a_1 and a_2 of the extended plane, consider the class $\mathfrak{M}(a_1, a_2) \equiv \mathfrak{M}(a_1, a_2; 0, |z| < 1)$ and its subclass $\mathfrak{M}_S(a_1, a_2)$ of systems of univalent functions. Let the systems $\tilde{f}_\nu(z)$, $\nu = 1, 2$, of the class $\mathfrak{M}_S(a_1, a_2)$ correspond, under the mappings $w = \tilde{f}_\nu(z)$, to the images $\widetilde{D}_\nu$ of the disk $|z| < 1$, and let some real function $\mathfrak{F}(w_1, w_2)$ be defined for $w_\nu \in D_\nu$, $\nu = 1, 2$, for all systems of functions of the class $\mathfrak{M}_S(a_1, a_2)$.

Lemma. If in the class $\mathfrak{M}_S(a_1, a_2)$, for any points z_1, z_2 of the disk $|z| < 1$, one of the estimates

$$M_1(|z_1|, |z_2|) \leq \mathfrak{F}(f_1(z_1), f_2(z_2)) \leq M_2(|z_1|, |z_2|),$$

is valid, where M_1 is a decreasing and M_2 an increasing function of each of its arguments, then it is valid also in the class $\mathfrak{M}(a_1, a_2)$ and can be attained in it only by univalent functions.

Theorem 3. In the class $\mathfrak{M}(0, \infty)$, for any points z_1, z_2 of the disk $|z| < 1$, we have*

$$|\log(1 - f_1(z_1)/f_2(z_2))| \leq -\frac{1}{2} \log(1 - |z_1|^2)(1 - |z_2|^2),$$

$$|f_1(z_1)/f_2(z_2)| \leq |z_1 z_2| \sqrt{(1 - |z_1|^2)(1 - |z_2|^2)}.$$

The estimates are sharp for arbitrary z_1 and z_2 with $|z_1| = |z_2| < 1$, and are attained only by univalent functions.

Theorem 4. If $f(z)$, $f(0) = 0$, regular in $|z| < 1$, and $F(\zeta)$, $F(\infty) = \infty$, meromorphic in $|\zeta| > 1$, are functions without common values, then for $|z| < 1$ and $|\zeta| > 1$ we have

$$\left|\frac{f^2(z)f'(z)f'(0)F'(\zeta)}{(f(z)-F(\zeta))^2F^2(\zeta)F'(\infty)}\right| \leq \frac{|z|^2}{|\zeta|^2(1-|z|^2)(|\zeta|^2-1)},$$

$$\left|f'(z)f'(0)F'(\zeta)/F^2(\zeta)F'(\infty)\right| \leq \frac{1}{(1-|z|^2)(|\zeta|^2-1)}.$$

The estimates are sharp for arbitrary z and ζ with $|z| = 1/|\zeta| < 1$, and are attained only by univalent functions.

In particular, from Theorems 3 and 4 there follows directly a number of known (2–5) sharp estimates for Bieberbach-Eilenberg functions and functions related to them.

Let B_q be the annulus $q < |z| < 1$. The following generalization (6) of Schwarz' s lemma to the case of an annulus is known. If $f(z) \in R_a(B)$, then for any $z_0 \in B_q$ we have $|f(z_0)| \leq |H(z_0, 1; a_0(z_0), a)|$, where $H \equiv H(z, 1; a_0(z_0), a)$ is uniquely determined by the conditions: H is regular in $\overline{B}_q$, has $z = a_0(z_0) = -qe^{i \arg z_0/|a|}$ and $z = a$ as its only and simple zeros in B_q, and on $|z| = q$ and $|z| = 1$, $|H| \equiv 1$, $H|_{z=1} = 1$; H is expressed explicitly in terms of

$$\theta(z) = \sum_{n=-\infty}^{+\infty} q^{n^2} z^n.$$

The extremal functions are $f(z) = \varepsilon H,\ |\varepsilon| = 1$.

Theorem 5. For B_q, $F(z, a) = \alpha H(z, 1; a_0(a), a)$, where

$$\alpha = \exp(-i \arg H'(a, 1; a_0(a), a)).$$

This theorem complements the Schwarz lemma just given and gives an explicit expression for the function $F(z, a)$ in the case of an annulus.

For brevity, put $H(\zeta, 1; a_0(z), a)|_{\zeta=z} = H(z, 1; a_0(z), a) = H(z)$. Let a_1 and a_2, $a_1 \neq a_2$, be given points of the extended plane.

Lemma. If in the class $\mathfrak{M}_S(a_1, a_2)$, for any points z_1, z_2 of the disk $|z| < 1$, one of the estimates

$$M_1(|z_1|, |z_2|) \leq \tilde{\mathfrak{F}}(\tilde{f}_1(z_1), \tilde{f}_2(z_2)) \leq M_2(|z_1|, |z_2|),$$

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* $\log(1 - f_1(z_1)/f_2(z_2))$ denotes the value of that branch, single-valued in the unit disk, of the multivalued function of z_1 or z_2 which vanishes, respectively, for $z_1 = 0$ or $z_2 = 0$.

where M_1 is decreasing and M_2 is increasing in each of their arguments, then in the class $\mathfrak{M}(a_1, a_2; a, B_q)$, for any points z_1, z_2 of the ring B_q, the corresponding estimate is valid

$$M_1(|H(z_1)|, |H(z_2)|) \leqslant \tilde{\mathfrak{F}}(f_1(z_1), f_2(z_2)) \leqslant M_2(|H(z_1)|, |H(z_2)|),$$

and in this class it can be attained only by systems of functions of the form $f_\nu(z) = \tilde{f}_\nu(\xi_\nu H(z, 1; a_0(z), a))$, $|\xi_\nu| = 1$, $\nu = 1, 2$.

Theorem 3′. In the class $\mathfrak{M}(0, \infty; a, B_q)$, for any points z_1, z_2 of the ring B_q, we have

$$|\log(1 - f_1(z_1)/f_2(z_2))| \leqslant -\frac{1}{2}\log(1 - |H(z_1)|^2)(1 - |H(z_2)|^2),$$

$$|f_1(z_1)/f_2(z_2)| \leqslant |H(z_1)H(z_2)|/\sqrt{(1 - |H(z_1)|^2)(1 - |H(z_2)|^2)}.$$

The estimates are sharp for arbitrary z_1 and z_2 such that $|H(z_1)| = |H(z_2)|$.

Denote by $C_a(B)$ the class of all functions $f(z)$, regular in the domain B, and satisfying the conditions: $f(a) = 0$, $a \in B$; $f(z_1)f(z_2) \neq 1$, $z_1, z_2 \in B$. In particular, the class C of Bieberbach–Eilenberg functions is the class $C_0(|z| < 1)$.

Theorem 6. For every given function $f(z) \in C_a(B)$ there exists a univalent function $\tilde{f}(\tau) \in C$ such that $f(z)$ does not assume in B those values which, for $|\tau| < 1$, are not assumed by $\tilde{f}(\tau)$.

Theorem 7. If $f(z) \in C_a(B)$, then in B we have the sharp estimate

$$|f'(z)| \leqslant |1 - f^2(z)|F'(z, z).$$

If $f(z) \in C_a(B_q)$, then in B_q we have the sharp estimates

$$|\log(1 - f^2(z))| \leqslant -\log(1 - |H(z)|^2), \qquad |f(z)| \leqslant |H(z)|/\sqrt{1 - |H(z)|^2}.$$

The lemmas stated above generalize to the case of any number of functions without common values, forming such systems of domains of values which have a filling. Consider the domain D of values of $f(z)$, regular in B. That one of the boundary continua of D each point of which can be joined with the point $w = \infty$ by a continuous curve not passing through points of D, we shall call the **outer boundary of the domain of values** of $f(z)$, or, briefly, the outer boundary of $f(z)$.

Theorem 8. If $f(z)$ is regular in B, $f(a) = 0$, $f'(a) = 1$, $a \in B$, then among the n points of the outer boundary of $f(z)$ nearest to $w = 0$, lying on n arbitrary rays emanating from $w=0$ at equal angles, the point farthest from $w=0$ is at a distance

$$d \geqslant 1/\sqrt[n]{4F'(a,a)}$$

from $w=0$, and equality holds only for

$$f(z)=F(z,a)/(1-\varepsilon F^n(z,a))^{2/n}, \qquad |\varepsilon|=1.$$

Similarly, Pic' s well-known theorem on bounded univalent functions in the unit disk carries over to the class $R_a(B)$.

Theorem 9. If $f(z)$ is regular in B_q, $f(a)=0$, $f'(a)=1$, $a\in B_q$, and d is the distance from the origin $w=0$ to the nearest point of the outer boundary of $f(z)$, then in B_q we have the sharp estimate

$$|f(z)| \leqslant 4d\,|H(z)|/(1-H(z))^2.$$

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Note: Figure translations are in progress. See original paper for figures.