PROPAGATION OF A HEAT WAVE CLOSE TO SPHERICAL

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Abstract

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The self-similar problem of heat propagation with thermal conductivity depending on temperature was first solved by Ya. B. Zel' dovich and A. S. Kompaneets ([1]). As applied to the theory of gas filtration in a porous medium, self-similar solutions of an equation analogous to the heat-conduction equation, independently of ([1]), were studied in detail by G. I. Barenblatt ([2]).

Of interest is the case when the form of the wave differs little from spherical, and the law of heat propagation is close to self-similar.

Suppose that at the initial instant of time a quantity of heat Q is released in a small volume (point). Consider a heat wave, for a power-law dependence of the coefficient of thermal conductivity on temperature, propagating in a medium at rest with variable density. We shall assume that the density differs little from a constant, and that the initial temperature of the medium is zero.

The equation of heat influx in a spherical coordinate system is written as follows:

$$cR\frac{\partial T}{\partial t} = \frac{\partial^2 T^k}{\partial r^2} + \frac{2}{r}\frac{\partial T^k}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T^k}{\partial \theta^2} + \frac{\operatorname{ctg}\theta}{r^2}\frac{\partial T^k}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T^k}{\partial \varphi^2}, \qquad (1)$$

where $c = \rho_* \alpha_0 k/\varkappa_0$ is a constant; $\varkappa = \varkappa_0 T^{k-1}$ and α_0 are the coefficients of thermal conductivity and heat capacity; $\rho_0(r,\theta,\varphi) = R\rho_*$; ρ_0 and ρ_* are the initial and characteristic densities. (If the heat capacity depends on temperature, $\alpha = \alpha_0 T^\omega$, then, introducing the variable $T' = \int_0^T \alpha_0 T^\omega\, dT$, we again obtain an equation of type (1) ([1]).)

Let us also write the condition of conservation of energy:

$$C\gamma(k) = \frac{Q}{\alpha_0\rho_*} = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0^{r_\phi} RT\, r^2\, dr = \text{const}. \qquad (2)$$

If the density is constant everywhere, then the problem is characterized only by two constants c and C, whose dimensions are expressed through the dimensions of length $[L]$, time $[t]$, and temperature $[T]$:

$$[c] = \frac{[T]^{k-1}[t]}{[L]^2}; \qquad [C] = [T][L]^3.$$

From the quantities r, t, c, and C one can form only one dimensionless combination

$$\xi = r t^{-\frac{1}{3k-1}} C^{-\frac{1}{3k-1}} c^{\frac{1}{3k-1}},$$

therefore such a problem is self-similar.

For variable density it is necessary also to take into account the dependence of the solutions on the angles and on time. Put

$$\rho_0 = \rho_* \left[1 + \sum_{\lambda} \sum_{n,m} Y_n^m(\theta, \varphi)(r\beta)^{(3k-1)\lambda}\right] =$$

$$= \rho_* \left[1 + \sum_{\lambda} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\frac{t}{\tau_\lambda}\right)^{\lambda} \xi^{(3k-1)\lambda} Y_n^m(\theta, \varphi)\right]. \tag{3}$$

Here

$$Y_n^m(\theta, \varphi) = \begin{cases} A_{nm} P_n^m(\cos\theta)\cos m\varphi, & m < 0, \\ B_{nm} P_n^m(\cos\theta)\sin m\varphi, & m > 0, \end{cases}$$

are spherical functions; A_{nm}, B_{nm} are constants; $\tau_\lambda = \dfrac{c}{C^{k-1}\beta}$ is the characteristic time. Introduce the dimensionless temperature

$$T = \left[cC^2 t^{-1}\right]^{3/(3k-1)} \left[f(\eta) + f_1\left(\eta, \theta, \varphi, \frac{t}{\tau_\lambda}\right)\right]. \tag{4}$$

Here

$$f(\eta) = \left[\frac{k-1}{2k(3k-1)}(1-\eta^2)\right]^{1/(k-1)}$$

corresponds to the self-similar solution, but is calculated not at the point ξ, but at the point $\eta = \xi/\xi_\phi$ separated from it. The value of η at the wave front is always equal to unity: $\eta_\phi = 1$.

We linearize the problem, regarding the quantities proportional to f_1 and $(t/\tau_\lambda)^\lambda$ as small. By virtue of the linearity of the theory it is sufficient to find the solution corresponding to one harmonic and to use the superposition principle.

The variables in the equation for small quantities can be separated if the solution is sought in the form (3)

$$f_1\left(\eta,\theta,\varphi,\frac{t}{\tau_\lambda}\right)=\left(\frac{t}{\tau_\lambda}\right)^\lambda H(\eta)Y_n^m(\theta,\varphi). \quad (5)$$

In an analogous form we shall also seek ξ_ϕ:

$$\xi_\phi=1+K_{nm}^\lambda\left(\frac{t}{\tau_\lambda}\right)^\lambda Y_n^m(\theta,\varphi). \quad (6)$$

Separating the variables and using the solution for $f(\eta)$, we obtain for $H(\eta)$ the linear ordinary differential equation of second order

$$\eta^2(1-\eta^2)\frac{d^2H}{d\eta^2}+\eta\left(2+\frac{8-6k}{k-1}\eta^2\right)\frac{dH}{d\eta}+$$

$$+H\left\{\eta^2\left[\frac{6-2\lambda(3k-1)}{k-1}-6+n(n+1)\right]-n(n+1)\right\}=$$

$$=\frac{2\eta^2}{k-1}f(\eta)\left[n(n+1)-6+\frac{4+2\lambda(3k-1)}{(k-1)(1-\eta^2)}\eta^2\right]K_{nm}^\lambda-$$

$$-\frac{2}{k-1}\eta^2 f(\eta)\left[3-\frac{2\eta^2}{(k-1)(1-\eta^2)}\right],\qquad n=0,1,2,... \quad (7)$$

As boundary conditions for this equation one may use the requirement that the temperature vanish at the wave front according to a prescribed law. In paper ([1]) it is shown that at the front of a heat wave of arbitrary shape the dimensionless temperature is proportional to $(1-\eta)^{1/(k-1)}$. Hence we arrive at the following condition for H as $\eta\to 1$:

$$H(\eta)=\varepsilon_0(1-\eta)^{1/(k-1)}[1+O(\eta)], \quad (8)$$

where ε_0 is determined from equation (7) and is equal to

$$\varepsilon_0=\frac{[2+\lambda(3k-1)]K_{nm}^\lambda+1}{k-1}\left[\frac{k-1}{k(3k-1)}\right]^{1/(k-1)}.$$

When (8) is satisfied, the physical requirements that the temperature and the heat flux vanish at the wave front are satisfied automatically. To find a particular solution one may choose a point η_*, close to $\eta=1$, determine from (8) $H(\eta_*)$ and $(dH/d\eta)_{\eta=\eta_*}$, and solve the Cauchy problem.

The solution of equation (7) may be sought as the sum $H = K^{\lambda}_{nm}H_0 + H_1$, where H_1 satisfies (7) and (8) for $K^{\lambda}_{nm} = 0$. The value of the constant K^{λ}_{nm} is determined from the condition that there be no heat flux in any direction at the center of the wave. Using (5), we write this requirement as

$$\lim_{\eta\to 0} \eta^2 \frac{dH}{d\eta} = 0. \tag{9}$$

The asymptotic solution of equation (7) without the right-hand side as $\eta \to 0$ is sought in power form:

$$H = \eta^{\beta_2}(K^{\lambda}_{nm}A_0 + A_1) + O(\eta^{\beta_2}). \tag{10}$$

After substituting (10) into (7), we find two roots $\beta_{1,2} = -1/2 \pm \sqrt{1/4 + n(n+1)}$. Since $\beta_2 < -1$ for $n > 0$, condition (9) gives

$$K^{\lambda}_{nm} = -A_1/A_0. \tag{11}$$

It follows that, in the first approximation, the wave front contains no harmonics other than that which enters the expansion for the density. (All $K^{\lambda}_{lq} = 0$, except $l = n,\ q = m$, since the corresponding $A_1 = 0$.) Since by choosing K^{λ}_{nm} from condition (11) one can eliminate the solution that grows without bound at the center, the linear theory is applicable to our problem throughout the entire region of propagation of the heat wave.

Let us determine the value of $\gamma(k)$ from consideration of the heat balance. We write (2) in dimensionless form, neglecting quantities of second order of smallness; we find:

$$\gamma(k) = 4\pi \int_0^1 \eta^2 f\, d\eta + \left(\frac{t}{\tau_\lambda}\right)^{\lambda} \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, Y^m_n(\theta,\varphi)\, d\theta \times$$

$$\times \int_0^1 \eta^2 \left[f\eta^{(3k-1)\lambda} + H + 3fK^{\lambda}_{nm}\right] d\eta. \tag{12}$$

The condition of conservation of energy requires that the right-hand side of (12) not depend on time. The case $\lambda = n = 0$, $(t/\tau_\lambda)^\lambda = \varepsilon$, corresponds simply to a renormalization of the front of the self-similar wave. For all $n \geq 1$ the energy-conservation condition is satisfied by virtue of the orthogonality of the spherical

functions. If $n = m = 0$, then the requirement of conservation of energy is satisfied only for a specific value of K^{λ}_{00}:

$$K_{00}^{\lambda} = -\frac{\displaystyle\int_0^1 \left[f\eta^{(3k-1)\lambda} + H_1\right]\eta^2\,d\eta}{\displaystyle\int_0^1 (3f + H_0)\eta^2\,d\eta}. \tag{13}$$

Substituting $f(\eta)$ into (12), we find that the values of $\gamma(k)$ for our problem and for the self-similar model problem coincide and are expressed in the same way through the gamma function:

$$\gamma(k) = 2\pi\left[\frac{k-1}{2k(3k-1)}\right]^{1/(k-1)}\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{k}{k-1}\right)\left[\Gamma\left(\frac{3}{2}+\frac{k}{k-1}\right)\right]^{-1}. \tag{14}$$

The condition of conservation of energy makes it possible to determine Q from the coordinates of the wave at a given instant of time. Integrating (6) over the solid angle, we obtain:

$$Q = \gamma(k)\rho_*\alpha_0 c^{1/(k-1)}t^{-1/(k-1)}\left[\frac{1}{4\pi}\int r\,\sin\theta\,d\theta\,d\varphi\right]^{(3k-1)/(k-1)}. \tag{15}$$

If $n = 0$, then the corresponding harmonic, determined from (13), must be excluded in this expression from r .

In a manner analogous to that described above, one can linearize other solutions [(2)] that are close to self-similar ones and find the subsequent approximations.

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CITED LITERATURE

[1] Ya. B. Zel' dovich, A. S. Kompaneets, in: Collection dedicated to the 70th anniversary of A. F. Ioffe, Moscow, 1950.
[2] G. I. Barenblatt, *Applied Mathematics and Mechanics*, **16**, no. 1 (1952).
[3] E. I. Andriankin, *Doklady Akademii Nauk SSSR*, **111**, no. 3 (1956).

Note: Figure translations are in progress. See original paper for figures.