ON PERIODIC SOLUTIONS OF A NONLINEAR EQUATION OF SELF-OSCILLATIONS

In the present article we study periodic solutions of a nonlinear equation of self-oscillations of the form:

1957

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. M. VOLOSOV

ON PERIODIC SOLUTIONS OF A NONLINEAR EQUATION OF SELF-OSCILLATIONS

(Presented by Academician I. G. Petrovskii on 17 I 1957)

§ 1. Formulation of the problem

In the present article we study periodic solutions of a nonlinear equation of self-oscillations of the form:

$$\ddot{x} + Q(x) = \varepsilon f(x, \dot{x}, \varepsilon), \tag{1}$$

where ε is a small parameter. Let $\operatorname{sign} Q = \operatorname{sign} x$ and

$$\int_0^{\pm\infty} Q(x)\, dx = \infty;$$

then the unperturbed equation

$$\ddot{x}_0 + Q(x_0) = 0 \tag{2}$$

has only periodic solutions. The perturbed equation (1) will have periodic solutions, generally speaking, not for all, but only for certain special initial conditions. In the present work formulas are derived for computing the principal characteristics of the periodic solutions of (1)—the amplitude and the period, and these quantities are expressed directly through the functions Q, f entering into (1). Nonperiodic solutions of (1), close in initial conditions to a certain periodic solution, as $t \to \infty$ may approach this periodic solution without bound or move away from it. In connection with this, a criterion for the stability of periodic solutions of (1) is given here, which is also determined only through the functions Q, f.

Suppose that Q and f are regular in $x, \dot{x}, \varepsilon$ for all values of $x, \dot{x}$ considered below and for all sufficiently small ε. We shall seek a periodic solution of (1), depending analytically on ε in some neighborhood of $\varepsilon = 0$, satisfying certain, as yet unknown, initial conditions of the form

$$x(0) = F_1(\varepsilon) = \sum_{n=0}^{\infty} F_{1n}\varepsilon^n, \quad \dot{x}(0) = 0.$$

Put $F_{10} > 0$, i.e. we shall measure t from a positive maximum of the solution $x(t, \varepsilon)$ of equation (1). It can be shown that a periodic solution (1) which does not pass, for $\varepsilon = 0$, into the trivial solution $x_0 = 0$ of equation (2), if it exists, for sufficiently small ε oscillates about the t-axis and has alternating positive maxima and negative minima, between which there is one zero each and no other extrema; moreover, between neighboring maximum and minimum the curve $x(t, \varepsilon)$ changes monotonically. Within one period there is then only one maximum and one minimum.

Let the minimum of the sought periodic solution (1) have magnitude

$$F_2(\varepsilon) = \sum_{n=0}^{\infty} F_{2n}\varepsilon^n.$$

For $\varepsilon = 0$ the periodic solution (1) passes into the solution x_0 of equation (2), satisfying the initial conditions $x_0(0) = F_{10}$,

$x_0(0) = 0$, and the minimum of x_0 is equal to F_{20}. Thus this solution x_0 of equation (2), for $\varepsilon \neq 0$, gives rise to a periodic solution of (1) with some period

$$T(\varepsilon) = \sum_{n=0}^{\infty} T_n\varepsilon^n,$$

where T_0 will be the period of x_0. The functions

$$F_1(\varepsilon) = \sum_{n=0}^{\infty} F_{1n}\varepsilon^n \quad \text{and} \quad F_2(\varepsilon) = \sum_{n=0}^{\infty} F_{2n}\varepsilon^n$$

are, respectively, the positive and negative amplitudes of the periodic solution (1). The values of the constant F_{10}, determining those solutions of equation (2) which, for $\varepsilon \neq 0$, give rise to periodic solutions (1), must be determined. The constants F_{1k} $(k > 0)$, F_{2j}, T_j $(j = 0, 1, 2, ...)$ will be functions of F_{10}, which are likewise to be determined.

For the solution of the problem posed, we use the method developed by the author in (^{5}); the advisability of applying it to the problem of periodic solutions of equation (1) was pointed out by L. E. Elsgolts.

§ 2. **Main results.**
1. For the quantities $F_{10} > 0$ and $F_{20} < 0$, which are the zero approximations of the amplitude of the periodic solution (1), a system of two finite equations of the form has been obtained:

$$A_1(F_{10}, F_{20}) \equiv \sum_{i,k=1}^{2} (-1)^{i+k+1} \int_0^{F_{k0}} f \left[x, (-1)^i \left(2 \int_0^{F_{k0}} Q(y)\, dy \right)^{1/2}, 0 \right] dx = 0,$$

$$A_2(F_{10}, F_{20}) \equiv \int_{F_{20}}^{F_{10}} Q(x)\, dx = 0. \tag{3}$$

In order that the solution x_0 of the unperturbed equation (2), satisfying the initial conditions $x_0(0) = F_{10}$, $\dot{x}_0(0) = 0$, where $F_{10} > 0$ and $F_{20} < 0$ are roots of (3), actually give rise, for sufficiently small $\varepsilon \neq 0$, to a periodic solution (1) depending analytically on ε, it is sufficient that Q and f be regular for the corresponding values of $x, \dot{x}$, and that for these values of F_{10}, F_{20} the Jacobian $\partial(A_1, A_2)/\partial(F_{10}, F_{20}) \neq 0$.

For the higher approximations of the amplitude F_{kj} ($k = 1, 2;\ j > 0$), systems of linear algebraic equations with determinant $\partial(A_1, A_2)/\partial(F_{10}, F_{20})$ have been obtained. We give here only the equations for the quantities F_{k1} ($k = 1, 2$)—the first approximation of the amplitude (εF_{k1} are corrections of order ε to the zero approximation of the amplitude F_{10}, F_{20}):

$$\frac{\partial A_1}{\partial F_{10}} F_{11} + \frac{\partial A_1}{\partial F_{20}} F_{21} = B_1(F_{10}, F_{20}),$$

$$Q(F_{10}) F_{11} - Q(F_{20}) F_{21} = B_2(F_{10}, F_{20}), \tag{4}$$

where

$$B_1(F_{10}, F_{20}) \equiv \sum_{i,k=1}^{2} \int_0^{F_{k0}} dx \left\{ (-1)^{k+1} f'_{\dot{x}} \left[x, (-1)^i \left(2 \int_x^{F_{k0}} Q(y)\, dy \right)^{1/2}, 0 \right] \times \right.$$

$$\times \left[\int_x^{F_{k0}} f \left[y, (-1)^i \left(2 \int_y^{F_{k0}} Q(z)\, dz \right)^{1/2}, 0 \right] dy \right] +$$

$$\left. + (-1)^{i+k} f'_{\varepsilon} \left[x, (-1)^i \left(2 \int_x^{F_{k0}} Q(y)\, dy \right)^{1/2}, 0 \right] \right\}.$$

$$B_2(F_{10}, F_{20}) = \frac{1}{2} \sum_{i,k=1}^{2} (-1)^{k+1} \int_0^{F_{k0}} \dot{f} \left[x, (-1)^i \left(2 \int_x^{F_{k0}} Q(y)\, dy \right)^{1/2}, 0 \right] dx,$$

where it is obvious that $\partial A_2/\partial F_{k0} = (-1)^{k+1}Q(F_{k0})$ $(k = 1, 2)$, and therefore the determinant of system (4) is indeed equal to $\partial(A_1, A_2)/\partial(F_{10}, F_{20})$.

The equations of the subsequent approximations F_{lj} $(l = 1, 2;\ j > 1)$ of the amplitude have a structure analogous to (4), and we do not write them out here.

2. For the period of the periodic solution (1) the expansion

$$T(\varepsilon) = \sum_{n=0}^{\infty} T_n \varepsilon^n,$$

has been obtained, where the coefficients T_n depend on F_{kj} $(k = 1, 2;\ j \leq n)$.

We give here only the expressions for the zeroth and first approximations of the period:

$$T_0 = 2\sum_{k=1}^{2}(-1)^{k+1}\int_0^{F_{k0}} dx \left(2\int_x^{F_{k0}} Q(y)\,dy\right)^{-1/2}, \tag{5}$$

$$\begin{aligned} T_1 = \sum_{k=1}^{2} \frac{\partial T_0}{\partial F_{k0}} F_{k1} + \sum_{i,k=1}^{2}(-1)^{k+1}\int_0^{F_{k0}} dx \left(2\int_x^{F_{k0}} Q(y)\,dy\right)^{-3/2} \\ \times \left\{\int_x^{F_{k0}} f\left[y, (-1)^i\left(2\int_y^{F_{k0}} Q(z)\,dz\right)^{1/2}, 0\right] dy\right\}. \end{aligned} \tag{6}$$

The remaining coefficients T_j $(j > 1)$ are expressed by analogous formulas, which we do not write out here.

3. The periodic solution (1), generated by the solution of the unperturbed equation (2) satisfying the initial conditions $x_0(0) = F_{10}$, $\dot{x}_0(0) = 0$, where $F_{10} > 0$ and $F_{20} < 0$ are the roots of (3), is stable for

$$\varepsilon\frac{\partial(A_1, A_2)}{\partial(F_{10}, F_{20})} < 0$$

and unstable for

$$\varepsilon\frac{\partial(A_1, A_2)}{\partial(F_{10}, F_{20})} > 0.$$

3. Résumé. Formulas of the form (3), (4), and (5), (6) have been derived for computing the amplitude and period of the periodic solutions (1), and a stability criterion for these solutions has also been given.

Periodic solutions of nonlinear systems have been studied in the works of a number of authors (^{1}–^{4}). In (^{1}–^{4}), theorems on the existence and stability of periodic solutions of systems of a general form were established, of which equation (1) is a special case.

In the present work, the more particular form of equation (1) (in comparison with the general systems considered in (^{1}–^{4})) made it possible to apply the methods of (^{5}), thanks to which it proved possible to derive finite formulas for the amplitude and period and to give stability criteria for periodic solutions expressed directly in terms of the functions Q, f.

I consider it my duty to express my deep gratitude to A. N. Tikhonov for valuable advice and guidance.

Moscow State University
named after M. V. Lomonosov

Received
15 I 1957

CITED LITERATURE

[1] I. G. Malkin, *Methods of Lyapunov and Poincaré in the Theory of Nonlinear Oscillations*, 1949.
[2] I. G. Malkin, *Theory of Stability of Motion*, 1952.
[3] A. M. Kats, *Prikl. matem. i mekh.*, 19, issue 1 (1955).
[4] I. G. Malkin, *Certain Problems in the Theory of Nonlinear Oscillations*, 1956.
[5] V. M. Volosov, *DAN*, 106, No. 1 (1956).

Note: Figure translations are in progress. See original paper for figures.