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# V. M. Tsodyks

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**Abstract**

**Full Text**

**V. M. Tsodyks**

**ON SETS OF POINTS WHERE THE DERIVATIVE IS RESPECTIVELY FINITE AND INFINITE**

*(Presented by Academician S. L. Sobolev, 15 I 1957)*

In the works of N. N. Luzin <sup>(1)</sup>, Z. S. Zagorskii <sup>(2)</sup>, and A. L. Brudno <sup>(3)</sup> it was shown that the set of points at which an infinite derivative exists is an  $F_{\sigma\delta}$  set of measure zero, while the set of points at which at least one derived number of a function of a real variable tends to infinity is a  $G_{\delta}$  set.

E. M. Landis <sup>(4)</sup>, for an arbitrary set  $E$  of type  $F_{\sigma\delta}$  and of measure zero, constructed a continuous function  $F(x)$  such that  $F'(x) = +\infty$  for  $x \in E$ ,  $\underline{F}(x) < +\infty$  for  $x \notin E$  (by  $\underline{F}$  is denoted the lower derived number of the function  $F$ ).

In the present note we briefly state a theorem giving a partial answer to the question of the relation between the set of points where the derivative is equal to infinity and the set of points where the derivative is finite <sup>(2)</sup>.

**Theorem.** *Let  $E$  be a set of type  $F_{\sigma\delta}$  of measure zero and  $N$  a set of type  $G_{\delta}$ , lying on the axis  $OX$ , with  $N \supset E$ . Then there exists a continuous increasing function  $F(x)$  such that  $F'(x) = +\infty$  for  $x \in E$ ,  $\underline{F}(x) < +\infty$  for  $x \notin E$ , and for  $x \in CN$ ,  $F'(x)$  exists and is finite.*

**Proof.**

1. Let

$$E = \prod_{n=1}^{\infty} E_n \quad \text{and} \quad E_n = \sum_{k=1}^{\infty} E_{nk},$$

where the  $E_{nk}$  are closed sets.

Let

$$N = \prod_{n=1}^{\infty} G_n,$$

where the  $G_n$  are open sets such that  $G_n \supset E$ .

We may assume that

$$\begin{aligned} \text{mes } G_n < \frac{1}{2^n}, \quad G_n \supset G_{n+1}, \quad E_n \subset G_n, \quad E_{nk} \subset E_{n,k+1}, \\ E_{n+1,k} \subset E_{nk}. \end{aligned}$$

Let a summable function

$$u(x) = \sum_{n=1}^{\infty} u_n(x)$$

be given, where  $u_n(x) = 1$  for  $x \in G_n$ , and  $u_n(x) = 0$  for  $x \notin G_n$ .

Place in  $E_n$  the set

$$e_n = \sum_{k=1}^{\infty} e_{nk},$$

where the  $e_{nk}$  are sets simultaneously of type  $F_\sigma$  and  $G_\delta$ , possessing the following properties:

- 1)  $e_{n,k+1} \supset e_{nk}$ ,  $E \cdot E_{nk} \subset e_{nk} \subset E_{nk}$ .
- 2) For every integer  $k \geq 2$  there exist six open sets  $g_{1k}^*$ ,  $g_{nk}^{*1}$ ,  $g_{nk}^{*2}$ , and  $g_{nk}$ ,  $g_{nk}^1$ ,  $g_{nk}^2$ , and for  $k = 1$  three open sets  $g_{n1}$ ,  $g_{n1}^1$ ,  $g_{n1}^2$  such that:

a)

$$G_n = g_{n1} \supset g_{n1}^1 \supset g_{n1}^2 \supset e_{n1};$$

b) if  $x_0 \in g_{n1}$ , then for every  $h$

$$\frac{\text{mes}(g_{n1}^1)^h}{h} \leq \frac{1}{2^n},$$

where

$$(g_{n1}^1)^h = g_{n1}^1 \cdot [x_0, x_0 + h]$$

(or  $g_{n1}^1 \cdot [x_0 + h, x_0]$ , if  $h < 0$ );

- c)  $g_{n,k-1}^2 - e_{n,k-1} = g_{nk}^* \supset g_{nk}^{*1} \supset g_{nk}^{*2} \supset g_{nk} \supset g_{nk}^1 \supset g_{nk}^2 \supset (e_{nk} - e_{n,k-1})$ , where  $k \geq 2$ ;  $g_{nk} \subset G_k$ ;
- d) the points  $Cg_{nk}$  are points of density for  $Cg_{nk}^1$ ; the points  $Cg_{nk}^1$  are points of density for  $Cg_{nk}^2$ ; the points  $Cg_{nk}^2$  are points of density for  $Cg_{nk}^{*1}$ ; the points  $Cg_{nk}^{*1}$  are points of density for  $Cg_{nk}^{*2}$ ;
- e) if  $x_0 \notin g_{nk}^{*2}$ , then for any  $h$

$$\frac{\int_{g_{nk}^h} u(\xi) d\xi}{h} < \frac{1}{2^k},$$

where  $g_{nk}^h = g_{nk} \cdot [x_0, x_0 + h]$  (or  $g_{nk} \cdot [x_0 + h, x_0]$ , if  $h < 0$ ).

Obviously,  $E \subset e_n \subset E_n$ ,  $\prod_{k=1}^{\infty} g_{nk} \subset N$ , and the points of the set  $e_n$  are points of density for the set

$$D_n = \sum_{k=1}^{\infty} (g_{nk}^2 - g_{n,k+1}^{*1}).$$

Let  $e_{n1} = \prod_{l=1}^{\infty} G_{n1}^{(l)}$ , where  $G_{n1}^{(l)}$  is an open set,  $G_{n1}^{(l+1)} \subset G_{n1}^{(l)} \subset g_{n1}$ ,  $\overline{G_{n1}^{(l+1)}} \subset G_{n1}^{(l)} + E_{n1}$ .

Let  $e_{nk} - e_{n,k-1} = \prod_{l=1}^{\infty} G_{nk}^{(l)}$  ( $k = 2, 3, \dots$ ), where  $G_{nk}^{(l)}$  is an open set,  $G_{nk}^{(l+1)} \subset G_{nk}^{(l)} \subset g_{nk}$ ,  $\overline{G_{nk}^{(l+1)}} \subset G_{nk}^{(l)} + E_{nk}$ .

Put  $G_n^{(l)} = \sum_{k=1}^{\infty} G_{nk}^{(l)}$  (obviously,  $e_n \subset G_n^{(l)} \subset G_n$ ) and construct, for each natural  $l$ , open sets  $A_{nl}, A_{nl}^1, A_{nl}^2$  such that:

- a)  $G_n^{(1)} \cdot G_1 = A_{n1} \supset A_{n1}^1 \supset A_{n1}^2 \supset E$ ;
- b)  $A_{n,l-1}^2 \cdot G_n^{(l)} \cdot G_l = A_{nl} \supset A_{nl}^1 \supset A_{nl}^2 \supset E$ , where  $l \geq 2$ ;
- c) the points  $CA_{nl}$  are points of density for  $CA_{nl}^1$ ; the points  $CA_{nl}^1$  are points of density for  $CA_{nl}^2$ .

2. We now construct auxiliary functions.

For each natural  $n$  we construct asymptotically continuous functions <sup>(5,6)</sup>

- 1)  $\theta_{nk}(x)$  ( $k = 1, 2, \dots$ ), where  $\theta_{nk}(x) = 0$  for  $x \in Cg_{nk}^1$ ;  $\theta_{nk}(x) = 1$  for  $x \in g_{nk}^2$ ;  $0 \leq \theta_{nk}(x) \leq 1$  for the remaining points.
- 2)  $\theta_{n,k+1}^*(x)$  ( $k = 1, 2, \dots$ ), where  $\theta_{n,k+1}^*(x) = 0$  for  $x \in Cg_{n,k+1}^{*1}$ ;  $\theta_{n,k+1}^*(x) = 1$  for  $x \in g_{n,k+1}^{*2}$ ;  $0 \leq \theta_{n,k+1}^*(x) \leq 1$  for the remaining points.
- 3)  $v_{nl}(x)$  ( $l = 1, 2, \dots$ ), where  $v_{nl}(x) = 0$  for  $x \in CA_{nl}^1$ ;  $v_{nl}(x) = 1$  for  $x \in A_{nl}^2$ ;  $0 \leq v_{nl}(x) \leq 1$  for the remaining points.

Put

$$\theta_n(x) = \sum_{k=1}^{\infty} (\theta_{nk}(x) - \theta_{n,k+1}^*(x)); \quad w_{nl}(x) = \min[\theta_n(x), v_{nl}(x)].$$

We construct a summable function

$$w_n(x) = \sum_{l=1}^{\infty} w_{nl}(x).$$

At the points  $CN$  the functions  $\theta_n(x), w_{nl}(x), w_n(x)$  are asymptotically continuous; for  $x \in A_{nl}^2 \cdot D_n$ ,  $w_n(x) \geq l$ .

Let, further,

$$W_n(x) = \int_0^x w_n(\xi) d\xi.$$

Then for  $x \notin E_n$  we have  $0 \leq W_n(x) < +\infty$ .

3. We proceed to the construction of the desired function. Put

$$w_n^*(x) = \min_{m \leq n} w_m(x); \quad w_n^{**}(x) = \max[0, w_n^*(x) - (n - 1)].$$

Let

$$f_n(x) = \min[1, w_n^{**}(x)].$$

At points of  $CN$  the functions  $f_n(x)$  are asymptotically continuous;  $f_n(x) \leq w_n(x)$ ; if  $x_0 \in E$ , then  $x_0$  is a point of density of the set on which  $f_n(x) = 1$ .

Now put

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad F(x) = \int_0^x f(\xi) d\xi.$$

Observe that

$$f(x) < w_n^*(x) + n, \quad F(x) \leq W_n(x) + n.$$

The increasing function  $F(x)$ , absolutely continuous on every interval of the  $OX$ -axis, satisfies the conditions of the theorem:

- I. If  $x_0 \in E$ , then  $F'(x_0) = +\infty$ .
- II. If  $x^* \notin E$ , then  $\overline{F}(x^*) < +\infty$ .
- III. If  $x \in CN$ , then  $F'(x)$  exists and is finite.

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## CITED LITERATURE

- <sup>1</sup> N. N. Luzin, Collected Works, 1, Publishing House of the Academy of Sciences of the USSR, 1953, pp. 5-24. <sup>2</sup> Z. S. Zagorskii, Mat. sbornik, 9(51), 3, 487 (1941). <sup>3</sup> A. L. Brudno, Mat. sbornik, 13(55), 1, 119 (1943). <sup>4</sup> E. M. Landis, DAN, 107, No. 2 (1956). <sup>5</sup> G. P. Tolstov, Mat. sbornik, 8(50), 1, 149 (1940). <sup>6</sup> V. S. Bogomolova, Mat. sbornik, 32, 1, 152 (1924).

*Note: Figure translations are in progress. See original paper for figures.*

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