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L. A. SAKHNOVICH

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Abstract

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L. A. SAKHNOVICH

ON THE REDUCTION OF NON-SELF-ADJOINT OPERATORS TO DIAGONAL FORM

(Presented by Academician M. V. Keldysh, 28 II 1957)

The questions of expansion in eigenfunctions and associated functions of various classes of non-self-adjoint operators with discrete spectrum have been studied by Birkhoff, Tamarkin, M. V. Keldysh, and others ($^{1-4}$). In the work of M. A. Naimark (^{5}), a spectral expansion was first given for a non-self-adjoint differential operator of the second order with continuous spectrum.

In the present paper we consider non-self-adjoint operators A of the class $i\Omega$ (^{6}) with continuous spectrum. Recall that a bounded operator A, defined in a Hilbert space H, belongs to the class $i\Omega$ if its "imaginary part"

$$\frac{A - A^*}{2i}$$

is a completely continuous operator with a convergent sum of the moduli of its eigenvalues. We find sufficient conditions under which the operator A can be reduced to diagonal form. The results obtained are used to investigate the behavior of $\|\psi(t)\|$ as $t \to \infty$, where $\psi(t)$ is a solution of the Schrödinger equation

$$ih\frac{\partial \psi(t)}{\partial t} = A\psi(t), \qquad \psi(0) = \psi_0 \quad (\psi_0 \in H).$$

1. As is known (^{6}), every operator A ($A \in i\Omega$) with continuous spectrum is unitarily equivalent (up to an additional component)* to the operator $A_1 f = g$:

$$g(x) = \alpha(x)f(x) + i\int_x^l f(t)\beta(t)J\,dt\,\beta(x) \tag{1}$$

$$(0 \leq x \leq l,\ f(x) \in L_r^2[0,l])^{**},$$

   where the matrix $\beta(x)$ $(0 \leq x \leq l)$ is nonnegative and $\operatorname{Sp}\beta(x) \equiv 1$, while the matrix J is diagonal, and its diagonal elements are equal either to $+1$ or to -1.

Thus, the question of the possibility of reducing the operator A ($A \in i\Omega$) to diagonal form will be resolved if the operator A_1 (1) is reduced to such a form.

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* The additional component of the operator A is the maximal invariant subspace on which the equality $A = A^*$ holds.

** Let E be a bounded measurable set. By the space $L_r^2[E]$ $(r \leq \infty)$ we shall understand the space of vector-functions

$$f(x) = [f_1(x), f_2(x), \dots, f_r(x)], \quad x \in E,$$

with scalar product defined as follows:

$$(f, g) = \sum_{i=1}^{r} \int_E f_i(x)\overline{g_i(x)}\, dx.$$

Let $H_1 = L_r^2[E]$. By the operator of multiplication by the independent variable Q_{H_1} in the space H_1 we shall mean the operator defined by the equality

$$Q_{H_1}\varphi(\sigma) = \sigma\varphi(\sigma), \qquad \varphi(\sigma) \in H_1. \tag{2}$$

Obviously, the operator Q_{H_1} may be regarded as a continual analogue of a diagonal matrix.

We shall say that an operator A is reduced to diagonal form if there exists a bounded operator B, mapping H_1 one-to-one onto H and satisfying the condition

$$B^{-1}AB = Q_{H_1}. \tag{3}$$

Let us note that not every operator of the class $i\Omega$ with continuous spectrum is reducible to diagonal form. Indeed, the operator (1) with $\alpha(x) \equiv 0$ cannot be reduced to diagonal form.

Suppose that the function $t = \alpha(x)$ $(0 \leq x \leq l)$ has no intervals of constancy. In addition, suppose that the function $\sigma(t) = x$ $(\alpha(0) \leq t \leq \alpha(l))$, inverse to the function $t = \alpha(x)$, has a uniformly bounded derivative $\sigma'(t) = p^2(t)$ $(\alpha(0) \leq t \leq \alpha(l))$. Under these assumptions the "triangular model" of the operator A may be transformed to the form

$$A_2 f = xf(x) + i\int_a^x f(t)\beta_1(t)J\, dt\, \beta_1(x) \qquad (a \leq x \leq b), \tag{4}$$

where $\beta_1(t) = p(t)\beta(\sigma(t))$, $a \leq t \leq b$, $a = \alpha(0)$, $b = \alpha(l)$.

Indeed, it is not difficult to see$^{(6)}$ that the characteristic matrix-functions of the operators A_2 and A_1 coincide and, consequently, these operators are unitarily equivalent (up to the supplementary component).

2. Let us consider first the case when the rank of the non-Hermitian part of the operator A_2 is equal to 1.

Theorem 1. *If* $p^2(x)$ $(a \leq x \leq b)$ *is a function of bounded variation and satisfies a Lipschitz condition* $(0 < \alpha \leq 1)$, *then the operator*

$$A_2 f = x f(x) + i p(x) \int_a^x f(t) p(t) j \, dt \qquad (a \leq x \leq b, \ j = \pm 1) \tag{5}$$

can be reduced to diagonal form.

The supplementary component of the operator (5) consists of those and only those vectors $f(x) \in L^2[a,b]$ for which the equality

$$f(x) p(x) \equiv 0 \tag{6}$$

holds.

Let us note that one can find an analytic expression for the operators B and B^{-1}. Let $p^2(x) \geq q > 0$ $(a \leq x \leq b)$; then the operator B reducing the operator A_2 to diagonal form is given by the formula

$$B\varphi = -\frac{j}{2\pi} \frac{d}{dx} \int_a^x \varphi(\sigma) \left\{ \exp[\pi p^2(\sigma)] - \exp[-\pi p^2(\sigma)] \right\}^{1/2} \times$$

$$\times \exp\left[-i \int_a^x \frac{p^2(s) j}{s - \sigma} \, ds \right] d\sigma \, p(x)^{-1}, \qquad (a \leq x \leq b), \tag{7}$$

where $\varphi(\sigma) \in L^2[a,b]$ (the integral with a prime is understood in the sense of the Cauchy principal value).

The operator B^{-1} is defined by the equality

$$B^{-1} f = \left\{ \int_a^\sigma \left[\frac{f(x)}{p(x)} \right]' \exp\left[i \int_a^x \frac{p^2(\nu)}{\nu - \sigma} \, d\nu \right] dx + \frac{f(a)}{p(a)} \right\} \left\{ \exp[\pi p^2(\sigma)] - \exp[-\pi p^2(\sigma)] \right\}^{1/2}$$

$$\left(a \leqslant \sigma \leqslant b, \quad \left[\frac{f(x)}{p(x)} \right]' \in L^2[a,b] \right). \tag{8}$$

Obviously, the functions $f(x)$ $(f(x), [f(x)/p(x)]' \in L^2[a,b])$ form a dense set in $L^2[a,b]$. Consequently, by formula (8) the bounded operator B^{-1} is completely defined.

Formulas (7) and (8) are preserved without essential changes also in the case when $p^2(x)$ vanishes at some set of points. In this case they give a reduction to diagonal form of the simple part* of the operator A_2. Let the operator A_2 have arbitrary rank of non-Hermiticity $J = \pm I$; then the following theorem is valid.

Theorem 2. Represent the matrix $\beta_1(x)$ in the form $\beta_1(x) \equiv p(x)\beta(x)$ $(p(x) \equiv \operatorname{Sp}\beta_1(x),\ a \leqslant x \leqslant b)$. Suppose that for every x only N eigenvalues $\lambda_1(x), \lambda_2(x), \ldots, \lambda_N(x)$ of the matrix $\beta(x)$ are nonzero and that, for some $\delta > 0$, the inequality

$$\lambda_i(x) \geqslant \delta \quad (a \leqslant x \leqslant b;\ i = 1, 2, \ldots, N)$$

is satisfied. If, moreover, for some K and for any $x_1, x_2 \in [a, b]$ the inequality

$$\left\|\beta_1^2(x_2) - \beta_1^2(x_1)\right\| < K|x_2 - x_1|,$$

holds, then the corresponding operator

$$A_2 f = xf(x) + i\int_a^x f(t)\beta_1(t)J\,dt\,\beta_1(x) \quad (a \leqslant x \leqslant b,\ J = \pm I) \tag{9}$$

can be reduced to diagonal form. The supplementary component of the operator A_2 (9) consists of those and only those vectors $f(x) \in L_r^2[a, b]$ for which the equality $f(x)\beta_1(x) \equiv 0$ holds.

3. Let an operator A be given in some space H. A subspace G is called **generating** if the linear closed span of the manifolds A^{nG} $(n = 0, 1, 2, \ldots)$ coincides with H. The **multiplicity** (or **total multiplicity**) of the spectrum of the operator A is the minimal dimension of the generating subspaces of this operator.

Theorem 3. If $\beta_1(x)$ satisfies the conditions of Theorem 2, then the multiplicity of the spectrum of the simple part of the corresponding operator A_2 (9) is equal to the maximal rank N of the matrix $\beta_1(x)$ $(a \leqslant x \leqslant b)$.

4. Using the preceding results, one can prove the following uniqueness theorem for systems of differential equations.

Theorem 4. Consider two systems

$$\frac{dW(x,\lambda)}{dx} = iW(x,\lambda)\frac{\beta_1^2(x)}{x-\lambda} \quad (a \leqslant x \leqslant b);$$

$$\frac{dW(x,\lambda)}{dx} = iW(x,\lambda)\frac{\beta_2^2(x)}{x-\lambda} \quad (a \leqslant x \leqslant b),$$

where $\beta_1(x)$ and $\beta_2(x)$ satisfy the conditions of Theorem 2.

If the Wronskians of these systems coincide $[W_1(b,\lambda) = W_2(b,\lambda) = W(\lambda)]$, then the relations

$$V\beta_1(x)V^{-1} = \beta_2(x) \quad (a \leqslant x \leqslant b); \qquad VW(\lambda)V^{-1} = W(\lambda),$$

hold, where V is a certain constant unitary matrix.

The results of §§ 2–4 also extend to operators of the form (4) with by an arbitrary matrix J, if it is additionally required that the matrix $\beta^2(x)J$ $(a \leqslant x \leqslant b)$ have no multiple eigenvalues.

5. Consider the Schrödinger equation

$$ih\frac{\partial\psi(t)}{dt} = A\psi, \qquad \psi(0) = \psi_0 \quad (\psi_0 \in H), \tag{10}$$

where A is an operator of class $i\Omega$ with continuous spectrum. From the relation

$$\frac{d\|\psi\|^2}{dt} = \frac{1}{h}\left(\frac{A - A^*}{i}\psi, \psi\right) \tag{11}$$

it follows that, under the condition

$$\frac{A - A^*}{i} < 0,$$

the norm of the solution $\|\psi(t)\|$ is a decreasing function of t.

Let ψ_j be a complete orthonormal system of elements of the space H; then the probability P (per unit time) of the decay of the system is expressed by the formula

$$P = -\sum_{j=1}^{\infty} \frac{d\|\psi_j(t)\|^2}{dt}\bigg|_{t=0} = \frac{1}{h}\left|\mathrm{Sp}\,\frac{A - A^*}{i}\right|. \tag{12}$$

For the class of operators considered by us, $P < \infty$. Since $\|\psi(t)\|^2$ is proportional to the number of particles in the system, the question arises under what conditions $\|\psi(t)\|^2$ tends to a number different from zero as $t \to \infty$. Since $\|\psi(t)\|^2$ is unchanged under unitary transformations of the space H, in solving this question one may assume that the operator A has been reduced to the "triangular form" (4).

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* The **simple part** of A_2 is the operator induced by means of A_2 on the subspace orthogonal to the supplementary component.

Theorem 5. If the operator A satisfies the conditions of Theorem 2, then, whatever the initial element ψ_0, the relation

$$m\|\psi_0\| \leqslant \|\psi(t)\| \leqslant M\|\psi_0\|, \qquad 0 \leqslant t < \infty, \tag{13}$$

holds, where M and m ($M > m > 0$) are constants independent of ψ_0.

Indeed, the solution $\psi(t)$ has the form

$$\psi(t) = B\left[e^{-i\frac{\sigma}{\hbar}t}\varphi_0(\sigma)\right], \tag{14}$$

where B is the operator reducing A to diagonal form; $\varphi_0(\sigma) = B^{-1}\psi_0$. From the boundedness of the operators B and B^{-1}, inequality (13) follows immediately.

6. Consider the equation

$$\frac{\partial^2\psi}{\partial t^2} + A\psi = 0, \qquad \psi(0) = \psi_0, \quad \left.\frac{\partial\psi}{\partial t}\right|_{t=0} = \psi_1 \quad (\psi_0, \psi_1 \in H), \tag{15}$$

where A is an operator of class $i\Omega$ with continuous spectrum. In addition, we shall assume that the spectrum of the operator A is contained in some interval $[c, d]$ $(0 < c < d)$.

Theorem 6. If the operator A satisfies the conditions of Theorem 2, then, whatever the initial elements ψ_0, ψ_1, the relation

$$m(\|\psi_0\| + \|\psi_1\|) \leqslant \|\psi(t)\| + \left\|\frac{\partial\psi}{\partial t}\right\| \leqslant M(\|\psi_0\| + \|\psi_1\|) \qquad (0 \leqslant t < \infty), \tag{16}$$

holds, where M and m ($M > m > 0$) are constants independent of ψ_0 and ψ_1.

Odessa Pedagogical Institute
named after K. D. Ushinsky

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Note: Figure translations are in progress. See original paper for figures.

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