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Abstract

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ON LOCAL NOMOGRAPHY

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The aim of the present work is to bring together the local theory of nomography ([1,2]) and investigations on the nomographability of functions as a whole. This aim is achieved by means of the invariant formulations obtained in our work of the theorems of M. A. Kreines and N. D. Eisenstadt.

For the function $z = f(x, y)$ we denote, following Gronwall ([3]):

$$M = -\frac{f_y}{f_x}, \qquad N = M_x + \frac{1}{M}M_y, \qquad \overline{N} = M_x - \frac{1}{M}M_y.$$

Definition. A *nomographic invariant* of a function $z = f(x, y)$, sufficiently smooth in a domain G of the plane xOy and such that f_x, f_y do not vanish in G, is a rational function $R = R(M, M_x, \dots, M_{y\dots y})$ of M and the partial derivatives of M, possessing the following property. Under any admissible transformation ([2])

$$x = x(X), \qquad y = y(Y), \qquad Z = Z(z) \tag{1}$$

the identity with respect to X, Y holds:

$$R(M^*, M^*_X, \dots, M^*_{Y\dots Y}) = \left(\frac{dx}{dX}\right)^{\alpha}\left(\frac{dy}{dY}\right)^{\beta} R(M, M_x, \dots, M_{y\dots y}),$$

where on the left the function R depends on M^*, computed for the function $Z = Z[f(x(X), y(Y))] = F(X, Y)$. The integers α, β are called the *weight* of the invariant with respect to x and y, respectively. Obviously, $M^* = -F_Y/F_X = My'/x'$, i.e. M is a nomographic invariant.

Lemma 1. The determinant of order k

$$W_k(M) = \begin{vmatrix} M & \dfrac{\partial M}{\partial x} & \dots & \dfrac{\partial^{k-1} M}{\partial x^{k-1}} \\ \dfrac{\partial M}{\partial y} & \dfrac{\partial^2 M}{\partial x\,\partial y} & \dots & \dfrac{\partial^k M}{\partial x^{k-1}\partial y} \\ \dots & \dots & \dots & \dots \\ \dfrac{\partial^{k-1} M}{\partial y^{k-1}} & \dfrac{\partial^k M}{\partial x\,\partial y^{k-1}} & \dots & \dfrac{\partial^{2k-2} M}{\partial x^{k-1}\partial y^{k-1}} \end{vmatrix} \tag{2}$$

provided that all the partial derivatives entering into it exist, is a nomographic invariant of the function $f(x, y)$ of weight $(k-3)k/2$ with respect to x and of weight $(k+1)k/2$ with respect to y.

Proof is easily carried out if transformation (1) is represented as the product of three admissible transformations:

$$
\begin{array}{llll}
x_1 = X(x), & y_1 = y, & z_1 = z; & \text{(I)} \\
x_2 = x_1, & y_2 = Y(y_1), & z_2 = z_1; & \text{(II)} \\
X = x_2, & Y = y_2, & Z = Z(z_2). & \text{(III)}
\end{array}
$$

and trace how the determinant (2) changes under each of these transformations.

If in Lemma 1 one sets $k = 2$, then one obtains the following lemma.

Lemma 2. The expression of Sen-Robert

$$P = \frac{\partial^2}{\partial x\, \partial y} \ln |M|$$

is a nomographic invariant of weight 1 in each of the variables.

Nomographic invariants of weight 0 in x and weight 6 in y are the two Bittner expressions ${}_1(M)$, ${}_2(M)$, whose vanishing, as is known ([4]), for $P \neq 0$ is a necessary and sufficient condition for nomographability by a Cauchy nomogram. Nomographic invariants of weight 0 in x and weight 1 in y are those occurring in (5),

$$\sigma_1 = M(\ln PM)_x + \left(\ln \frac{M}{P}\right)_y, \qquad \sigma_2 = M(\ln PM)_x - \left(\ln \frac{M}{P}\right)_y .$$

A generalization of the invariants σ_1, σ_2 is, under the assumption of sufficient smoothness of M, the expression

$$S(k, m) = \overline{M}(\overline{M} \dots (\overline{M}^{k+1}(M(M \dots (MP)_{x\dots})_x)_x)y \dots)_y,$$

where differentiation with respect to x is performed k times, differentiation with respect to y is performed m times, $\overline{M} = 1/M$, and at the place where the differentiations with respect to x pass to differentiations with respect to y, $(\overline{M})^{k+1}$ is inserted, which is retained even if there are no differentiations with respect to y. $S(k, m)$ is a nomographic invariant of weight $k + m + 2$ in x.
Definition. The **exceptional points** of the function $z = f(x, y)$ are called the points of the domain G satisfying the system of equations

$$P = 0, \qquad P_x = 0, \qquad P_y = 0, \qquad P_{xx}M^2 + P_{xy}M + P_{yy} = 0 \tag{3}$$

and such that at any of these points

$$D = P_{xxy}M + P_{xyy} + 3P_{xy}\overline{N} + P_{xx}M_y + \frac{M_x}{M}P_{yy} \tag{4}$$

does not vanish.

Theorem 1. The function $z = f(x, y)$ is nomographable with accuracy up to small terms of the 6th order in a neighborhood of any point of the domain G, and with accuracy up to small terms of the 7th order in a neighborhood of any non-exceptional point of this domain.

The **proof** of this theorem follows from the sequence of lemmas. It is known ([2]) that the coefficients of an admissible transformation (1) can be chosen so that, in the variables X, Y, Z, the function $z = f(x, y)$ is represented in the form

$$Z = F(X, Y) = X + Y + XY(X - Y)(q_{00} + q_{10}X + q_{01}Y + ... + q_{04}Y^4) + + o(\rho^7), \tag{5}$$

where $\rho = \sqrt{X^2 + Y^2}$.

Lemma 3. The coefficients q_{ik} are integral rational functions of the nomographic invariant $P^*(X, Y)$ and of the derivatives of P^*, computed at the point $X = Y = 0$.

Let us give the first formulas:

$$q_{00} = -\frac{1}{4}P^*, \qquad q_{10} = -\frac{5}{42}P^*_X - \frac{1}{21}P^*_Y, \qquad q_{01} = -\frac{1}{21}P^*_X - \frac{5}{42}P^*_Y.$$

Lemma 4. A sufficiently smooth function $f(x, y)$ in the domain G is locally nomographable with accuracy up to small terms of the 7th order at any point of the domain G, if and only if in this domain the system of equations (3) is inconsistent.

Proof. If one takes into account the relation between $P(x, y)$ and $P^*(X, Y)$, between their derivatives, and also Lemma 3, then the inconsistency of the system (3) at an arbitrary point $(x_0, y_0) \in G$ is equivalent to the fact that among the numbers

$q_{00}, q_{10}, q_{01}, q_{20} + q_{02} - q_{11}$ are nonzero, which, as is known ([2]), is the condition for nomographability up to infinitesimals of the 7th order.

To complete the proof of Theorem 1 it remains to show that at those points of the domain G at which equations (3) are satisfied, the expression D reduces to $2(q_{30} - q_{03}) - 5(q_{21} - q_{12})$. The proof of this fact follows from the following. At a point $(x_0, y_0) \in G$ satisfying equations (3), one may write

$$2(q_{30} - q_{03}) - 5(q_{21} - q_{12}) = \frac{1}{24}\frac{\partial^2}{\partial X\,\partial Y}\,[P^*_X - P^*_Y].$$

Moreover, for $X = Y = 0$ we have $M^* = -1$, $M^*_X = M^*_Y = N^* = M^*_{XX} = M^*_{XY} = M^*_{YY} = 0$, whence it follows that $D^*(X,Y) = -P^*_{XXY} + P^*_{XYY}$, and consequently $D^* = x_0'^2 y_0'^3 \cdot D$, which proves Theorem 1.

Theorem 2. *The coefficients q_{ik} of the function (5) are expressed nomographically invariantly in terms of the nomographic invariants M, P of the function $z = f(x,y)$ and their partial derivatives with respect to x, y.*

The proof of the theorem is based on Lemma 3 and the application of the invariant $S(k,m)$.

We give the first formulas:

$$q_{00} = \frac{1}{4} MP\, x_0'^2, \qquad q_{10} = \left[\frac{5}{42} S(1,0) - \frac{1}{21} S(0,1)\right] x_0'^3,$$

$$q_{01} = \left[\frac{1}{21} S(1,0) - \frac{5}{42} S(0,1)\right] x_0'^3.$$

It is known [(2)] that local nomographing with accuracy up to infinitesimals of the k-th order is effected by constructing, in local coordinates, a Massau determinant $\Delta[X,Y,Z]$ such that $\Delta[X,Y,F(X,Y)] = O(\rho^k)$. If one takes into account that the coefficients of this determinant $a_i, b_i, c_i, \alpha_i, \beta_i, \gamma_i$ are expressed through q_{ik}, then the following theorem is thereby proved.

Theorem 3. *The coefficients of the expansions in the rows of the local Massau determinant are expressed nomographically invariantly in terms of the invariants M, P of the function $z = f(x,y)$ and their partial derivatives.*

The proof follows from the formulas of the paper [(2)], if $\alpha_1, \alpha_2, \beta_2, \gamma_3$ are regarded as nomographic invariants with sums of weights, respectively, 1, 2, 2, 3, which can be done by virtue of their arbitrariness.

Thus a possibility has been obtained of composing the local Massau determinant $\Delta[X,Y,Z]$ directly from the function $z = f(x,y)$, without first reducing it to the special form (5).

We now consider an estimate of the principal term of the error when nomographing up to infinitesimals of the 6th order.

Theorem 4. *If $f(x,y)$ is a sufficiently smooth function, given in a domain G, such that in G its first derivatives nowhere vanish, then the minimum error $|f(x,y) - n(x,y)|$, where $n(x,y)$ is a nomographable function approximating $f(x,y)$ in a neighborhood of the point $(x_0, y_0) \in G$ with accuracy up to infinitesimals of the 6th order, satisfies the inequality*

$$|f(x,y) - n(x,y)| < \frac{1}{576}\, \delta\rho^6 + o(\rho^6),$$

where $\delta = \sup_{G_1} |f_x(x,y) D|$, and the set G_1 is the set of points of the domain G at which

$$P = 0, \qquad P_x = 0, \qquad P_y = 0, \qquad P_{xx}M^2 + P_{xy}M + P_{yy} = 0, \qquad D \neq 0.$$

Proof. On passing to local coordinates we shall have

$$\Delta[X, Y, F(X, Y)] = \Delta_6^* + o_1(\rho^6),$$

whence it follows that

$$|F(X, Y) - N(X, Y)| = \Delta_6^* + o_2(\rho^6).$$

Here

$$\Delta_6^* = \sum_{i+j=6} K_{ij} X^{iY^{j;}}$$

the coefficients K_{ij} are regarded as functions of $\alpha_1, \alpha_2, \dots, \gamma_5$ and are connected by the relation

$$K_{42} + K_{24} - \frac{3}{2} K_{33} = -\frac{1}{2} A,$$

where A is the left-hand side of the known [(2)] condition (A).

The values of the coefficients $K_{ij} = K_{ij}^*$ can be chosen so that the form $\Delta^*(X, Y)$ realizes the minimax, i.e.

$$\lambda = \min_K \max |\Delta_6^*|,$$

where K is the circle of radius ρ_0, and the minimum is taken with respect to variations of $\alpha_1, \alpha_2, \dots, \gamma_5$. Indeed,

if $f(x, y)$ is not nomographable up to infinitesimals of order 7, then $K_{42} + K_{24} - {}^3/_2 K_{33} = -{}^1/_{48} D^* \neq 0$, and it is required to find a form $\Delta_\delta^*(X, Y)$ that deviates least from zero in the Chebyshev sense, under an additional condition relating its coefficients. The existence of such a form was proved in [6]. Let us estimate the magnitude λ. From the preceding it follows that $\lambda \leq D^* R$, where R is the smallest of the moduli ${}^1/_{48} X^4 Y^2$, ${}^1/_{72} X^3 Y^3$, ${}^1/_{48} X^2 Y^4$ in the closed disk of radius ρ_0. Since $R = \max_{\rho' \leq \rho_0} {}^1/_{72} X^3 Y^3 = {}^1/_{576} \rho^6$, the inequality $\lambda \leq \dfrac{|D^*|}{576} \rho^6$ is valid. Returning to the old variables x, y, z, we obtain the proof of the theorem.

We shall now use the arbitrary parameters $\alpha_1, \alpha_2, \beta_2, \gamma_3$ to simplify the form of the local nomogram.

Theorem 5. *A sufficiently smooth function $z = f(x, y)$ at the point x_0, y_0, such that*

$$f'_x(x_0, y_0) \neq 0, \qquad f'_y(x_0, y_0) \neq 0, \qquad P(x_0, y_0) \neq 0,$$

is represented in a neighborhood of the point x_0, y_0 with accuracy up to infinitesimals of order 5 by a Cauchy nomogram.

Proof. We require that the supports of the scales X and Y be straight lines. Then the equation of the support for the scale X will be $\eta = k\xi + d$, and for the scale Y, $\eta = k_1 \xi + d_1$. Consequently, between the coefficients of the expansions in the rows of the determinant the relations

$$\alpha_i = k a_i, \qquad \beta_i = k_1 b_i, \qquad i \leq 4. \tag{6}$$

must hold.

Taking in each of these systems the first 3 equations and eliminating from them k and k_1, we obtain a system of 4 equations with 4 unknowns $\alpha_1, \alpha_2, \beta_2, \gamma_3$. If $q_{00} = -{}^1/_4 P(x_0, y_0) x_0' y_0' \neq 0$, then this system is compatible and has a unique solution. At the same time we have $k = k_1 = \alpha_1$. Computing a_4 and b_4 by the known formulas (2), but now with the already chosen $\alpha_1, \alpha_2, \beta_2, \gamma_3$, we find α_4 and β_4 from formulas (6). If one takes into account that the α_4 and β_4 thus obtained do not affect the error up to infinitesimals of order 5, then the theorem is proved.

Theorem 6. *If $P(x_0, y_0) = 0$, but $P(x, y) \neq 0$ in every neighborhood of x_0, y_0, then nomographing up to infinitesimals of order 5 by a Cauchy nomogram is possible in this case if and only if, at the point x_0, y_0, the Bitner invariants $B_1(M)$, $B_2(M)$ vanish.*

Proof is carried out easily if one takes into account that, for $P(x_0, y_0) = -4q_{00} = 0$, at the point x_0, y_0 we have $B_1(M) = 14(q_{10} - q_{01})(4q_{01} - 10q_{10})$, $B_2(M) = 14(q_{10} - q_{01})(10q_{01} - 4q_{10})$.

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Note: Figure translations are in progress. See original paper for figures.