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Abstract

Full Text

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ON A PROBLEM IN FILTRATION THEORY

(Presented by Academician I. G. Petrovskii, 9 III 1957)

In filtration theory, under an elastic-plastic regime ^(1,2), the following problem arises. Find a function p that satisfies the equation $\partial p/\partial t = a_1^2 \partial^2 p/\partial x^2$ for $\partial p/\partial t \geq 0$, the equation $\partial p/\partial t = a_2^2 \partial^2 p/\partial x^2$ for $\partial p/\partial t < 0$, and the conditions:

$$p(x, 0) = p_0(x) \quad (0 < x < \infty), \quad \left. \frac{\partial p}{\partial x} \right|_{x=0} = 0 \quad (t > 0). \quad (1)$$

Below a solution of this problem is given.

Instead of the two indicated equations for p , we shall consider the single equation equivalent to them

$$\alpha \frac{\partial p}{\partial t} + \beta \left| \frac{\partial p}{\partial t} \right| = \frac{\partial^2 p}{\partial x^2}, \quad (2)$$

where $\alpha + \beta = 1/a_1^2$, $\alpha - \beta = 1/a_2^2$. We note that $\alpha > 0$ and $|\beta| < \alpha$. In addition, we replace the conditions (1) for p by one initial condition

$$p(x, 0) = \varphi(x) \quad (-\infty < x < \infty), \quad (3)$$

where $\varphi(x)$ is an even function of x , equal to $p_0(x)$ for $x > 0$. We shall study the problem (2), (3) in the strip $R_T(-\infty < x < \infty, 0 \leq t \leq T)$ of arbitrary fixed width T .

Theorem 1. Let $\varphi(x)$ be bounded on the whole line and have three generalized (in the sense of Sobolev ⁽³⁾) derivatives belonging to $L_2(-\infty, \infty)$. Then there exists a continuous solution $p(x, t)$, bounded in R_T , of equation (2) under condition (3), having continuous derivatives $\partial p/\partial t$, $\partial p/\partial x$, $\partial^2 p/\partial x^2$ from $L_2(R_T)$ and the generalized derivative $\partial^2 p/\partial x \partial t \in L_2(R_T)$.

We give a brief proof. Consider the function $v(x, t) = \partial p/\partial t$. For $v(x, t)$ we have the equation

$$\frac{\partial(\alpha v + \beta|v|)}{\partial t} = \frac{\partial^2 v}{\partial x^2} \quad (4)$$

and the initial condition

$$v(x, 0) = \psi(x), \tag{5}$$

where

$$\psi(x) = \frac{\alpha\varphi'' - \beta|\varphi''|}{\alpha^2 - \beta^2}.$$

It is clear that $\psi(x) \in L_2(-\infty, \infty)$ and that there exists a generalized derivative $\psi'(x) \in L_2(-\infty, \infty)$. Introduce a quantity K such that

$$\int_{-\infty}^{\infty} [\psi(x)]^2 dx < K$$

and

$$\int_{-\infty}^{\infty} [\psi'(x)]^2 dx < K.$$

We shall find in the rectangle $Q_l (|x| \leq l, 0 \leq t \leq T)$ a continuous function $v(x, t)$ that has the generalized derivatives entering the equation, satisfies almost everywhere in Q_l equation (4) and the conditions

$$v(l, t) = v(-l, t) = 0, \quad v(x, 0) = \psi_1(x). \tag{6}$$

Here $\psi_1(x)$ is continuous, has a first generalized derivative, $\psi_1(x) = \psi(x)$ for $|x| \leq l - 1$, and $\psi_1(x) = 0$ for $|x| \geq l$. It is clear that $\psi_1(x)$ can be chosen so that

$$\int_{-l}^l [\psi_1(x)]^2 dx < K_1 \quad \text{and} \quad \int_{-l}^l [\psi_1'(x)]^2 dx < K_1,$$

where K_1 does not depend on l .

To construct the solution, we replace equation (4) in Q_l by the difference-differential equation, according to the Rothe method ⁽⁴⁾,

$$L_h v_h = \alpha \frac{v_h^n - v_h^{n-1}}{h} + \beta \frac{|v_h^n| - |v_h^{n-1}|}{h} - \frac{d^2 v_h^n}{dx^2} = 0. \tag{7}$$

Put $v_h^0 = \psi_1(x)$. Then, successively on each line $t_n = nh$, we determine $v_h^n(x)$ as a solution of the ordinary differential equation (7) under the boundary conditions $v_h^n(-l) = v_h^n(l) = 0$. The existence of such a solution is proved, for example, in ⁽⁵⁾.

Let h run through the sequence $h_i = \frac{T}{2^i}$. We shall find some estimates independent of h .

Consider the expression

$$\begin{aligned} h \sum_{n=1}^N \int_{-l}^l L_{hv} h \frac{v_h^n - v_h^{n-1}}{h} dx &= h(\alpha - |\beta|) \sum_{n=1}^N \int_{-l}^l \left(\frac{v_h^n - v_h^{n-1}}{h} \right)^2 dx + \\ &+ |\beta| \sum_{n=1}^N \int_{-l}^l \frac{[v_h^n - v_h^{n-1} + \text{sign } \beta (|v_h^n| - |v_h^{n-1}|)](v_h^n - v_h^{n-1})}{h} dx - \\ &- \sum_{n=1}^N \int_{-l}^l \frac{d^2 v_h^n}{dx^2} (v_h^n - v_h^{n-1}) dx = 0; \end{aligned} \quad (8)$$

here $N \leq T/h$.

After integration by parts we obtain

$$\frac{1}{2} \int_{-l}^l \left(\frac{dv_h^N}{dx} \right)^2 dx + h(\alpha - |\beta|) \sum_{n=1}^N \int_{-l}^l \left(\frac{v_h^n - v_h^{n-1}}{h} \right)^2 dx < \frac{1}{2} \int_{-l}^l [\psi_1'(x)]^2 dx = \frac{1}{2} K_1.$$

Since N is arbitrary, the last inequality means that

$$h \sum_{n=1}^{T/h} \int_{-l}^l \left(\frac{dv_h^n}{dx} \right)^2 dx + h \sum_{n=1}^{T/h} \int_{-l}^l \left(\frac{v_h^n - v_h^{n-1}}{h} \right)^2 dx < C_0, \quad (9)$$

where C_0 is a constant independent of h and l .

Denote by $\bar{v}_h(x, t)$ the function in Q_l which for $t = nh$ is equal to $v_h^n(x)$ and is linear in t for $(n-1)h < t < nh$. From (9) it is not difficult to obtain

$$\left\| \frac{\partial^2 \bar{v}_h}{\partial x^2} \right\|_{L_2(Q_l)} + \|\bar{v}_h\|_{W_2^{(1)}(Q_l)} < C_1, \quad (10)$$

where C_1 does not depend on h and l .

It follows from (10) that there exists a subsequence h_j such that $\{\bar{v}_{h_j}\}$ converges strongly in $L_2(Q_l)$ and weakly in $W_2^{(1)}(Q_l)$ to $v(x, t) \in W_2^{(1)}(Q_l)$, while $\{\partial^2 \bar{v}_{h_j} / \partial x^2\}$ converges weakly to $\partial^2 v / \partial x^2 \in L_2(Q_l)$ (3).

It can be proved that $v(x, t)$ almost everywhere satisfies equation (4), is continuous, and satisfies the initial and boundary conditions.

The solution of the Cauchy problem for equation (4) will be found as the limit of the solutions $v_l(x, t)$ of the first boundary-value problem for an infinitely expanding sequence of rectangles Q_l ($l = 1, 2, \dots$).

Set

$$p(x, t) = \int_0^t v(x, \tau) d\tau + \varphi(x).$$

It can be proved that $p(x, t)$ is continuous, bounded in R_T , satisfies the equation, and has the derivatives indicated in the theorem.

Lemma. If $p_1(x, t)$ and $p_2(x, t)$ are two solutions of equation (2), with $p_1(x, 0) = \varphi_1(x)$, $p_2(x, 0) = \varphi_2(x)$, and if there exist continuous derivatives $\partial p_i / \partial t$, $\partial p_i / \partial x$, $\partial^2 p_i / \partial x^2 \in L_2(R_T)$ and generalized derivatives $\partial^2 p_i / \partial x \partial t \in L_2(R_T)$ ($i = 1, 2$), then the inequality

$$\left\| \frac{\partial(p_1 - p_2)}{\partial x} \right\|_{L_2(R_T)}^2 + \left\| \frac{\partial(p_1 - p_2)}{\partial t} \right\|_{L_2(R_T)}^2 < C \int_{-\infty}^{\infty} (\varphi_1' - \varphi_2')^2 dx, \quad (11)$$

holds, where $C = \text{const}$ is independent of φ_1 and φ_2 .

Theorem 2. If $\varphi(x)$ is bounded on the whole line and has a generalized derivative $\varphi'(x) \in L_2(-\infty, \infty)$, then there exists a unique continuous solution $p(x, t)$, bounded in R_T , of equation (2) under condition (3), having, for $t > 0$, continuous derivatives $\partial p / \partial t$, $\partial p / \partial x$, and $\partial^2 p / \partial x^2$.

For the proof one must consider a sequence of functions $\varphi_k(x)$ with the following properties:

- a) $\varphi_k(x)$ has three generalized derivatives in $L_2(-\infty, \infty)$;
- b) $|\varphi_k(x)| < M$, where M does not depend on k ;
- c) $\varphi_k'(x) \rightarrow \varphi'(x)$ in $L_2(-\infty, \infty)$ as $k \rightarrow \infty$, and for every x_0 , $\varphi_k(x_0) \rightarrow \varphi(x_0)$.

Let $p_k(x, t)$ be the solution of equation (2), constructed in Theorem 1, under the condition $p_k(x, 0) = \varphi_k(x)$. Applying estimate (11) to any two functions of the sequence $\{p_k(x, t)\}$, it is not difficult to obtain that $\{p_k(x, t)\}$ converges strongly in $W_2^{(1)}(\Omega)$ to $p(x, t) \in W_2^{(1)}(\Omega)$ (here Ω is an arbitrary bounded domain in R_T).

It is clear that $p(x, t)$ satisfies equation (2). It can be proved that the function $p(x, t)$ is continuous, bounded in R_T , and has, for $t > 0$, continuous derivatives $\partial p / \partial t$, $\partial p / \partial x$, and $\partial^2 p / \partial x^2$; for $t = 0$, $p(x, 0) = \varphi(x)$.

The uniqueness of the solution with the indicated properties can be proved analogously to the way this is proved for the heat-conduction equation (7).

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