SOME CRITERIA FOR NONOSCILLATION OF A FOURTH-ORDER DIFFERENTIAL EQUATION

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Abstract

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SOME CRITERIA FOR NONOSCILLATION OF A FOURTH-ORDER DIFFERENTIAL EQUATION

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In the present note, in a certain sense, the results obtained in article (1) for a second-order differential equation are generalized to the fourth-order differential equation

$$\frac{d^4y}{dx^4} + \frac{d}{dx}\left[a(x)y'\right] + b(x)y = 0, \qquad x_0 \leq x < \infty \tag{1}$$

(the functions $a(x)$ and $b(x)$ are assumed to be real and continuous for $x \geq x_0$).

Following Sternberg ([2]), we shall call a fourth-order equation **nonoscillatory** if, beginning with some sufficiently large x_0, every nontrivial solution of it has, for $x \geq x_0$, no more than one double zero. Correspondingly, we shall call a fourth-order equation **oscillatory** if, for every finite x_0, there exists a nontrivial solution of this equation having, for $x \geq x_0$, more than one double zero. The point $x = x_1$ is called a **double zero** of the solution $y(x)$ if $y(x_1) = y'(x_1) = 0$.

In obtaining the nonoscillation criteria formulated below for equation (1), a lemma is used which is analogous to Sturm' s comparison theorem for a second-order differential equation.

Lemma 1. *If $a_1(x) \leq a_2(x)$, $b_1(x) \geq b_2(x)$, beginning with some sufficiently large x, then from the nonoscillation of the equation*

$$\frac{d^4y}{dy^4} + \frac{d}{dx}\left[a_2(x)y'\right] + b_2(x)y = 0$$

there follows the nonoscillation of the equation

$$\frac{d^4y}{dx^4} + \frac{d}{dx}\left[a_1(x)y'\right] + b_1(x)y = 0.$$

The proof of the lemma is based on Sternberg' s criterion ([2]). From Lemma 1 there immediately follows the following simple criterion for nonoscillation of equation (1):

If, beginning with some sufficiently large x, $a(x) \leq 0$, $b(x) \geq 0$, then equation (1) is nonoscillatory.

It is known that the differential equation $y'' + \frac{\alpha}{x^2} y = 0$ is oscillatory for $\alpha > 1/4$ and nonoscillatory for $\alpha \leq 1/4$; therefore, in article (1), when studying the oscillation of a second-order differential equation, the equation

$$y'' + \frac{1}{4x^2} y = 0.$$

Study of the fourth-order equation

$$\frac{d^4y}{dx^4} + \frac{d}{dx}\left(\frac{\alpha}{x^2} y'\right) + \frac{\beta}{x^4} y = 0$$

shows that this equation is nonoscillatory for $\beta \geq \omega(\alpha)$ and oscillatory for $\beta < \omega(\alpha)$, where

$$\omega(\alpha) = \frac{9}{4}\alpha - \frac{9}{16} \quad \text{for } \alpha \leq \frac{5}{2}; \qquad \omega(\alpha) = \frac{1}{4}(\alpha + 2)^2 \quad \text{for } \alpha \geq \frac{5}{2}.$$

In connection with this, when studying the oscillation of equation (1), we shall take, as the comparison equation, the differential equation

$$\frac{d^4y}{dx^4} + \frac{d}{dx}\left(\frac{\alpha}{x^2}\frac{dy}{dx}\right) + \frac{\omega(\alpha)}{x^4}\, y = 0, \tag{2}$$

which is nonoscillatory for every finite α.

In accordance with Lemma 1, the following criterion for nonoscillation of equation (1) holds.

Theorem 1. *If there exists an α such that, beginning with some sufficiently large x,*

$$a(x) \leq \frac{\alpha}{x^2}, \qquad b(x) \geq \frac{\omega(\alpha)}{x^4},$$

then equation (1) is nonoscillatory.

To establish a more refined criterion for nonoscillation of equation (1), we have considered the equation

$$\frac{d^4y}{dx^4} + \frac{d}{dx}\left[\left(\frac{\alpha}{x^2} + \varphi(x)\right) y'\right] + \left[\frac{\omega(\alpha)}{x^4} + \psi(x)\right] y = 0. \tag{3}$$

If

$$\int^{\infty} x \ln x \, |\varphi(x)| \, dx < \infty, \qquad \int^{\infty} x^3 \ln x \, |\psi(x)| \, dx < \infty, \tag{4}$$

then, as the study of the asymptotic behavior of solutions of the differential equation (3) shows, this equation is nonoscillatory for $\alpha \neq \frac{5}{2}$. Next put

$$\varphi(x) = \max\left\{a(x) - \frac{\alpha}{x^2}, 0\right\}, \qquad \psi(x) = \min\left\{b(x) - \frac{\omega(\alpha)}{x^4}, 0\right\}. \tag{5}$$

If these functions satisfy conditions (4), then for $\alpha \neq \frac{5}{2}$ the differential equation (3) is nonoscillatory and, consequently, in accordance with Lemma 1, equation (1) is also nonoscillatory, since for the functions $\varphi(x)$ and $\psi(x)$ defined by formulas (5) the inequalities

$$a(x) \leq \frac{\alpha}{x^2} + \varphi(x), \qquad b(x) \geq \frac{\omega(\alpha)}{x^4} + \psi(x)$$

hold.

We thus arrive at the following criterion for nonoscillation of equation (1).

Theorem 2. *If, for some finite $\alpha \neq \frac{5}{2}$,*

$$\int^{\infty} x \ln x \max\left\{a(x) - \frac{\alpha}{x^2}, 0\right\} dx < \infty,$$

$$\int^{\infty} x^3 \ln x \left|\min\left\{b(x) - \frac{\omega(\alpha)}{x^4}, 0\right\}\right| dx < \infty,$$

then equation (1) is nonoscillatory.

Remark. The differential equation (1) is also nonoscillatory if

$$\int^{\infty} x \ln^3 x \max\left\{a(x) - \frac{5}{2x^2}, 0\right\} dx < \infty,$$

$$\int^{\infty} x^3 \ln^3 x \left|\min\left\{b(x) - \frac{81}{16x^4}, 0\right\}\right| dx < \infty$$

(this criterion corresponds to the case $\alpha = 5/2$).

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References

[1] L. D. Nikolenko, *Ukr. Math. Zh.*, **7**, 127 (1955).
[2] R. Sternberg, *Duke Math. J.*, **19**, 311 (1952).

Note: Figure translations are in progress. See original paper for figures.