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Abstract

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ON THE MAXIMUM OF THE CONFORMAL RADIUS OF A FUNDAMENTAL DOMAIN OF A DOUBLY PERIODIC GROUP

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Let $\{D\}$ be a family of simply connected domains D of the w-plane, containing the point $w = 0$ and having the following properties: 1) the domain D contains no points congruent with respect to the group T_n of transformations $w' = w + n_1\omega_1 + n_2\omega_2$, where ω_1 and ω_2 are constants whose ratio is not real, while n_1 and n_2 are arbitrary integers; 2) the domain D contains none of the given system of finitely many points $a_1, \dots, a_m$ and no points congruent to them with respect to the group T_n.

Among all domains $\{D\}$ determine the domain that has the greatest conformal radius.

Denote by $S_a(\omega_1, \omega_2)$ the class of functions

$$w = f(z) = \sum_{n=1}^{\infty} c_n z^n, \tag{1}$$

regular in the disk $|z| < 1$ and univalently mapping this disk onto domains of the family $\{D\}$. The stated problem reduces to determining the maximum of $|f'(0)| = |c_1|$ in the class $S_a(\omega_1, \omega_2)$.

Theorem 1. If the function $w = f(z) \in S_a(\omega_1, \omega_2)$ gives the maximum of the functional $|f'(0)|$, then it satisfies the differential equation

$$\frac{1}{z^2 {w'}^2} = A_0 + \sum_{i=1}^{m} A_i \zeta(w - a_i) + A_{m+1}\zeta(w) + \wp(w) \tag{2}$$

where A_i $(i = 0, 1, \dots, m+1)$ are constants, with

$$\sum_{i=1}^{m+1} A_i = 0;$$

$\zeta(w)$, $\wp(w)$ are the Weierstrass functions constructed on the periods ω_1 and ω_2, and it maps the disk $|z| < 1$ onto a domain D having the following properties:

1) The domain D, containing the point $w = 0$, is a simply connected fundamental domain S_0 of the group T_n with cuts. Its boundary consists of a finite number of analytic arcs, pairwise congruent with respect to the group T_n, and of piecewise-analytic cuts issuing from the boundary of S_0 and ending at the points a_i $(i = 1, \dots, m)$ or at points congruent to them.

2) To each pair of congruent arcs of the boundary S_0 and to the simple arcs of the cuts there correspond, on the circle $|z| = 1$, by means of $w = f(z)$, two arcs of equal length.

Proof. We first derive a variational formula for functions of the class $S_a(\omega_1, \omega_2)$.

Consider the function

$$w^* = \Phi(w, |h|) = w + h\,[q_1(w) - q_1(0)] \prod_{i=1}^{m} \frac{\wp(w) - \wp(a_i)}{\wp(w) - b_i}, \tag{3}$$

where b_i are arbitrary constants; $q_1(w) = \zeta(w - w_{2m+1}) - \zeta(w - w_{2m+2})$, where w_{2m+1} and w_{2m+2} are distinct from the roots w_k $(k = 1, \dots, 2m)$ of the equations $\wp(w) = b_i$ $(i = 1, \dots, m)$, located in the fundamental parallelogram. As in ([1]), it is shown that if $f(z) \in S_a(\omega_1, \omega_2)$ and the image of the disk $|z| < 1$ under the mapping $w = f(z)$ covers all the points w_k $(k = 1, \dots, 2m+2)$, then the function $F(z, |h|) = \Phi(f(z), |h|)$, for sufficiently small $|h|$, will be regular and univalent in some annulus $r < |z| < 1$.

Applying the theorem of G. M. Goluzin ([2], Theorem 1), we obtain a function $f^*(z) \in S_a(\omega_1, \omega_2)$:

$$\begin{aligned} f^*(z) &= f(z) + h[q_1(f(z)) - q_1(0)] \prod_{i=1}^{m} \frac{\wp(f(z)) - \wp(a_i)}{\wp(f(z)) - b_i} \\ &\quad - hzf'(z) \sum_{k=1}^{2m+2} \frac{\beta_k}{z - z_k} + \bar{h} z^2 f'(z) \sum_{k=1}^{2m+2} \frac{\bar{\beta}_k}{1 - \bar{z}_k z} + O(|h|^2), \end{aligned} \tag{4}$$

where

$$\beta_k = \begin{cases} \dfrac{(-1)^{k+1}}{z_k {f'}^2(z_k)} \displaystyle\prod_{i=1}^{m} \frac{\wp(f(z_k)) - \wp(a_i)}{\wp(f(z_k)) - b_i}, & w_k = f(z_k) \quad (k = 2m+1,\ 2m+2); \\[2ex] \dfrac{q_1(f(z_k)) - q_1(0)}{z_k {f'}^2(z_k) \wp'(f(z_k))} \dfrac{\displaystyle\prod_{i=1}^{m} (\wp(f(z_k)) - \wp(a_i))}{\displaystyle\prod_{\substack{i=1 \\ (\wp(f(z_k)) \neq b_i)}}^{m} (\wp(f(z_k)) - b_i)}, & (k = 1, \dots, 2m),\ |z_k| < 1. \end{cases}$$

Next, let $w = f(z)$ be one of the extremal functions, and let D be the corresponding extremal domain. From the Lindelöf principle it follows that D is a certain simply connected fundamental domain S_0 of the group T_n, containing the point $w = 0$, with cuts running from the points $a_1, \dots, a_m$ to the boundary of S_0. Applying to the function $f(z)$ the variational formula (4) and taking into account the arbitrariness of $\arg h$, we obtain from the extremality condition for $f(z)$ the equality

$$q_1'(0) + \sum_{k=1}^{2m+2} \frac{\beta_k}{z_k} = 0. \tag{5}$$

Fixing $z_1, \dots, z_{2m}, z_{2m+2}$ and regarding z_{2m+1} as the variable z, we obtain an equation from which we conclude that the expression $\frac{1}{z^2 {w'}^2}$ is an elliptic function of the variable w. Using the known expression of elliptic functions in terms of the Weierstrass functions $\wp(w)$ and $\zeta(w)$, we arrive at the differential equation

$$\frac{1}{z^2 {w'}^2} = A_0 + \sum_{i=1}^{m} [A_i \zeta(w - a_i) + B_i \zeta(w + a_i)] + A_{m+1} \zeta(w) + \wp(w). \tag{6}$$

It is not difficult to show that all the coefficients $B_i = 0$.

Thus, equation (2) is established.

Assertions 1) and 2) of the theorem follow from equation (2).

Slightly modifying the proof of M. A. Lavrent' ev' s uniqueness theorem ([3]), Theorem 4; see also ([4]), p. 66), we obtain:

Theorem 2. *A domain D possessing properties 1) and 2) of Theorem 1 is unique.*

Relying on this theorem, in some particular cases one can find extremal domains by conjecture.

With the aid of the differential equation (2) the following theorem is proved.

Theorem 3. *The interior angles of the extremal domain D, situated at congruent vertices, are equal.*

Consider the net obtained as a result of tiling the plane w by the extremal domain D and by the domains congruent to it with respect to the group T_n. From Theorem 3 it follows:

Corollary. *The angles between two neighboring arcs issuing from a vertex of the net are equal.*

Fig. 1

In what follows we restrict ourselves to the case of real invariants g_2 and g_3 of the function $\wp(w)$.

Put $\Delta = g_2^3 - 27g_3^2$. Three cases are possible ((5), pp. 143–149):

1) $\Delta > 0$. In this case one period ω_1 is real, and the other ω_2 is purely imaginary.
2) $\Delta < 0$. In this case the periods ω_1 and ω_2 are complex conjugate numbers.
3) $\Delta = 0$. In this case: a) if $g_2 = 0$, then one period is finite and the other is equal to ∞; b) if $g_2 = g_3 = 0$, then $\omega_1 = \omega_2 = \infty$.

In case 3 b) the right-hand side of equation (2) is a rational function of w, for which the point $w = \infty$ is a zero of multiplicity not less than three. By integration we obtain the Laurent formula ((3), p. 180), and for $m = 1$ the Koebe function.

Case 3 a) reduces to the preceding case by mapping a simply connected vertical strip with slits onto the plane with slits.

For $\Delta \neq 0$ we consider the problem of the maximum of $|f'(0)|$ for the class $S(\omega_1, \omega_2)$ of functions $f(z)$ of the form (1), regular in the disk $|z| < 1$ and mapping it univalently onto domains not containing points congruent with respect to the group T_n. The extremal function in this case satisfies the differential equation

$$\frac{1}{z^2 {w'}^2} = A_0 + \wp(w), \tag{7}$$

where $A_0 = -\wp(w_0)$, and w_0 is a vertex of the extremal domain D. From (7) it follows that $c_{2k} = 0$ $(k = 1, 2, ...)$, whence we conclude that the points $\pm\frac{1}{2}\omega_1$ and $\pm\frac{1}{2}\omega_2$ lie on the boundary of D.

If $\Delta > 0$, then by Theorem 2 we conclude that the extremal domain will be a rectangle with center at the point $w = 0$ and with sides parallel to the coordinate axes, of lengths respectively ω_1 and $|\omega_2|$.

Finally, if $\Delta < 0$, then, by integrating equation (7), we find for the inverse function the expression

$$\ln(\overline{z_0} z) = \int_{w_0}^{w} \sqrt{\wp(w) - \wp(w_0)}\, dw, \tag{8}$$

where z_0 is a point of the circle $|z| = 1$ which is mapped to the vertex w_0, $w_0 = f(z_0)$. To determine the form of the extremal domain, we note that, when the point w lies on its boundary, the right-hand side of (8) must be equal to a purely imaginary number. This will occur only when, for these same values of w, the argument of the expression under the integral sign on the right-hand side of (8) is equal to $\pi/2$. In Fig. 1 the boundary of the extremal domain for values

$$\tau = \arg \frac{\omega_2}{\omega_1} \quad \text{from } 0 \text{ to } \frac{\pi}{2}$$

is represented schematically by solid lines.

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Note: Figure translations are in progress. See original paper for figures.