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Abstract

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MATHEMATICS

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ON A CERTAIN ANALOGUE OF THE THEORY OF EXTENSIONS OF HERMITIAN OPERATORS AND A NONSYMMETRIC ONE-DIMENSIONAL BOUNDARY-VALUE PROBLEM ON THE HALF-AXIS

(Presented by Academician A. N. Kolmogorov on 11 II 1957)

1. Let H be a Hilbert space; let J be some conjugation operator in H (i.e., an operator defined everywhere in H, satisfying the conditions $(Jf, Jg) = (g, f)$ and $J^2 f = f$ for any f and g in H (^{1}).

Definition 1. A linear operator acting in H is called J-symmetric if

$$(Af, Jg) = (f, JAg) \tag{1}$$

for all f and g in the domain of definition D_A of the operator A.

An example of a J-symmetric operator in $\mathcal{L}^2(0, \infty)$ is, in particular, the differential operator L' generated by the operation

$$l[y] = \sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k} \left[p_{n-k}(x) \frac{d^k y}{dx^k} \right] \tag{2}$$

on finite functions $\varphi(x)$, absolutely continuous together with their derivatives up to order $(2n-1)$, satisfying the condition $\varphi(0) = \varphi'(0) = \ldots = \varphi^{(2n-1)}(0) = 0$. Here it is assumed that the coefficients of the operation (2) are complex-valued functions satisfying the usual smoothness conditions; here and below, by the operator J in the space $\mathcal{L}^2(0, \infty)$ is meant the operator of complex conjugation, defined by the equality $J[f(x)] = \overline{f(x)}$ for any function $f(x)$ from $\mathcal{L}^2(0, \infty)$.

If the domain of definition D_A of a J-symmetric operator A is dense in H, then the operator A^* exists, and from (1) there follows the relation

$$JAJ \subset A^*. \tag{3}$$

Definition 2. A J-symmetric operator A with domain of definition dense in H is called J-self-adjoint if

$$JAJ = A^*.$$

An example of a J-self-adjoint operator in $L^2(0, \infty)$ is, in particular, the nonself-adjoint differential operator of second order investigated in the fundamental paper of M. A. Naimark (^{2}).

It follows from (3) that every J-symmetric operator with domain of definition dense in H admits a closure. In particular, the operator L' admits a closure, which we shall denote by L and call the differential operator with minimal domain of definition generated by the operation (2).

Let us note that a symmetric operator may in special cases turn out to be J-symmetric, and a self-adjoint operator J-self-adjoint—

(for example, the operator L in the case of real coefficients of the differential operation (2) and its real$^{(4)}$ self-adjoint extensions).

Definition 3$^{(3)}$. A linear operator A acting in H is called **dissipative** if

$$\operatorname{Im}(Af, f) \geqslant 0 \tag{4}$$

for every $f \in D_A$.

The operator L constructed above will be dissipative under the condition

$$\operatorname{Im} p_k(x) \geqslant 0 \qquad (k = 0, 1, \dots, n;\ 0 \leqslant x < \infty). \tag{5}$$

If A is a closed dissipative operator and $\operatorname{Im}\lambda < 0$, then the resolvent $R_\lambda = (A - \lambda I)^{-1}$ exists and $\|R_\lambda\| \leqslant |\operatorname{Im}\lambda|^{-1}$; moreover, the linear manifold $\Delta_\lambda = (A - \lambda I)D_A$ is closed, and the dimension m of its orthogonal complement does not depend on λ. This dimension m will be called the **defect number** of the operator A.

Theorem 1. *In order that a dissipative J-symmetric operator with domain dense in H be J-self-adjoint, it is necessary and sufficient that its defect number be equal to zero.*

With the aid of the pair of formulas

$$Af + if = g, \qquad Af - if = Vg$$

we introduce the Cayley transform V of a J-symmetric operator A with domain dense in H and defect number $m \geqslant 0$.

From (1) and (4) it follows that V is a J-symmetric operator in H, defined on the subspace $\Delta_{-i} \subset H$ ($\operatorname{def} \Delta_{-i} = m$), and $\|V\| \leqslant 1$.

If $\widetilde{V}$ is a J-symmetric extension of the operator V and $\|\widetilde{V}\| \leqslant 1$, then from the equality $\widetilde{V}h = h$ it follows that $h = 0$, and the operator $\widetilde{A}$, defined by the pair of formulas

$$f = \frac{1}{2i}(g - \widetilde{V}g), \qquad \widetilde{A}f = \frac{1}{2}(g + \widetilde{V}g),$$

is a J-symmetric dissipative extension of the operator A.

Modifying the method of M. G. Krein(5) for extending bounded symmetric operators with non-dense domain in H, we obtain a J-symmetric extension $\widetilde{V}$ of the operator V to all of H, satisfying the condition $\|\widetilde{V}\| \leqslant 1$. Hence Theorem 2 follows.

Theorem 2. *Every J-symmetric dissipative operator A with domain dense in H admits an extension to a J-self-adjoint dissipative operator $\widetilde{A}$.*

Let us note that from the J-self-adjointness of the operator A there follows the J-self-adjointness of its resolvent R_λ at regular points λ, which is essential for the proof of Theorem 5 (see below).

2. From what has been set forth, with the aid of the methods applied in (4), we obtain a series of theorems generalizing the known propositions (1,4) on differential operators of the form (2) with real coefficients to the case of complex-valued coefficients with nonnegative imaginary part.

Theorem 3. *If conditions (5) are fulfilled, then the number m of linearly independent solutions of the equation*

$$\sum_{k=0}^{n} (-1)^k \frac{d^k}{dx^k}\left[p_{n-k}(x)\frac{d^k y}{dx^k}\right] = \lambda y \qquad (\operatorname{Im}\lambda < 0),$$

belonging to $\mathscr{L}^2(0,\infty)$, does not depend on λ and satisfies the inequality $m \geqslant n$.

Theorem 4. The operator L with minimal domain of definition, generated by operation (2) with coefficients satisfying conditions (5), can be extended to a J-self-adjoint dissipative differential operator.

Theorem 5. The resolvent R_λ of any J-self-adjoint dissipative extension of the operator L for $\operatorname{Im}\lambda < 0$ is a bounded integral operator defined on all of $\mathscr{L}^2(0,\infty)$,

$$R_\lambda g = \int_0^\infty \Gamma(x, s; \lambda)\, g(s)\, ds$$

with kernel

$$\Gamma(x, s; \lambda) = \Gamma(s, x; \lambda).$$

For $m < 2n$ the resolvent kernel satisfies the condition

$$\int_0^\infty |\Gamma(x, s; \lambda)|^2\, ds < \infty.$$

For $m = 2n$,

$$\int_0^\infty \int_0^\infty |\Gamma(x, s; \lambda)|^2\, dx\, ds < \infty;$$

so that in this case the resolvent R_λ is a completely continuous operator.

For $n = 1$, Theorems 3-5 give a generalization of the well-known results of H. Weyl (5) to a differential equation of the form

$$-y'' + q(x)y - \lambda y = g(x) \qquad (0 \leqslant x < \infty),$$

where $q(x)$ is a complex-valued function summable on every finite interval, with nonnegative imaginary part $\operatorname{Im} q(x)$.

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