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Abstract

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ON INFINITE-DIMENSIONAL CANTOR MANIFOLDS

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1. As is known (1), an n-dimensional compactum (n a natural number) is called a **Cantor manifold of dimension** n if it cannot be separated by any closed subset of dimension $\leq n-2$ (i.e., the complement of every closed subset of dimension $\leq n-2$ is always connected; the dimension of the empty set is taken to be -1).

A compactum of dimension n is an n-dimensional Cantor manifold if and only if it is impossible to decompose this compactum into two closed subsets (proper ones) whose intersection has dimension $\leq n-2$.

Every n-dimensional compactum contains a Cantor manifold of the same dimension. This theorem was proved by Hurewicz (2) and, independently of him, by me (3).

2. Let us generalize the notion of a Cantor manifold to the case of infinite-dimensional compacta as follows.

Definition. An infinite-dimensional compactum is called an **infinite-dimensional Cantor manifold** if it cannot be separated by any finite-dimensional closed subset.

An infinite-dimensional compactum will be an infinite-dimensional Cantor manifold if and only if it is impossible to decompose this compactum into two closed subsets (proper ones) whose intersection is finite-dimensional.

It is not hard to see that, for example, the Hilbert parallelepiped is an infinite-dimensional Cantor manifold.

Examples show that an infinite-dimensional compactum may contain no infinite-dimensional Cantor manifold. Thus, for example, in the Hilbert parallelepiped one may take an infinite sequence of pairwise disjoint closed balls whose centers converge to some point, whose radii tend to zero, and whose dimensions increase without bound. The union of these balls with the limiting point forms an infinite-dimensional compactum containing no infinite-dimensional Cantor manifold (moreover, all components of this compactum are finite-dimensional, and it has not a single point at which it would be infinite-dimensional).

3. In 1926 I posed the following problem (4):

Does every infinite-dimensional compactum contain finite-dimensional compacta of arbitrarily large dimension?

This problem has not yet been solved. Only the existence of compacta of dimensions 1 and 2 was found (by Hemmert (6)).

In the present paper the following theorem is proved, which has some relation both to this problem and to the question of the existence of an infinite-dimensional Cantor manifold in an infinite-dimensional compactum.

Theorem. Whatever infinite-dimensional compactum R may be, one may assert that:

a) either R contains finite-dimensional compacta of arbitrarily large dimension;

b) or R contains an infinite-dimensional Cantor manifold.

Statements a) and b) do not exclude one another. The question whether every infinite-dimensional Cantor manifold contains finite-dimensional compacta of arbitrarily large dimension remains open.

4. The proof of the theorem is based on certain properties of Urysohn widths (1).

Let F be a closed set lying in some compact metric space. Denote by d_kF the lower bound of all numbers $\varepsilon > 0$ such that there exists a covering of F by a finite number of closed subsets of diameter less than ε, having the property that no point belongs to more than k of these subsets.

The Urysohn widths d_{kF} form a nonincreasing sequence of nonnegative numbers:

$$d_1F \geq d_2F \geq \cdots \geq d_{kF} \geq d_{k+1}F \geq \cdots$$

Of special importance for n-dimensional closed sets is the width d_{nF} (which P. S. Urysohn called the coefficient of compactness for F). F has dimension n if and only if $d_{nF} > 0$, while $d_{n+1}F = 0$.

If F is infinite-dimensional, then all $d_{kF} > 0$ and $\lim_{k\to\infty} d_{kF} = 0$.

Thus, if $d_{mF} > 0$ (m a natural number), then $\dim F \geq m$ (infinite dimension is regarded as greater than any finite dimension).

P. S. Alexandrov established (5) a close connection between Urysohn widths and ε-maps and ε-shifts of compacta into polyhedra of the corresponding dimension.

5. Of special significance for the proof of the theorem is the following property of Urysohn widths, proved by me (3).

Lemma. Let the common part of two closed sets A and B, lying in a compact metric space, have dimension $\leq n-2$ (n a natural number). In that case at least one of the widths d_{nA}, d_{nB} is equal to $d_n(A \cup B)$.

The proof of this proposition at the time required overcoming considerable constructive difficulties. Let us note only that no restrictions are imposed here on the dimensions of A and B; they may also be infinite-dimensional.

Moreover, if one takes a natural number $m > n$, then, a fortiori, at least one of the widths d_{mA}, d_{mB} is equal to $d_m(A \cup B)$. Indeed, if $\dim(A \cap B) \leq n-2$, then, a fortiori, the inequality $\dim(A \cap B) \leq m-2$ is satisfied.

6. We now prove the theorem. It is necessary to show that if there exists an infinite-dimensional compactum which does not contain finite-dimensional compacta of arbitrarily large dimension, then it contains an infinite-dimensional Cantor manifold. We shall carry out the proof, relying on the lemma, by means of transfinite induction.

Thus, let R be an infinite-dimensional compact metric space all of whose finite-dimensional subcompacta have dimension $\leq k$, where k is some fixed nonnegative integer.

Take the number $m = k + 2$. Since R is infinite-dimensional, $d_{mR} = d > 0$.

Put $R_1 = R$. Suppose that closed sets R_β have already been defined for all ordinal numbers $\beta < \alpha$ (where α is some number $< \Omega$) so that the following requirements are fulfilled:

1°. $R_\beta \supset R_{\beta+1}$ $(\beta + 1 < \alpha)$;

2°. $R_\gamma = \bigcap\limits_{\delta<\gamma} R_\delta$ (whatever ordinal number of the second kind $\gamma < \alpha$);

3°. $d_m R_\beta = d$.

We define R_α in the following way. If α is of the first kind, then either $R_{\alpha-1}$ is not split by any finite-dimensional closed subset (of dimension $\leqslant k$, since, by hypothesis, the compactum R contains no finite-dimensional closed subsets of dimension $> k$), or it is split.

In the first case we put $R_\alpha = R_{\alpha-1}$.

In the second case we have $R_{\alpha-1} = A \cup B$, where A and B are two closed proper subsets of $R_{\alpha-1}$ such that $\dim(A \cap B) \leqslant k$. By the lemma, at least one of the closed sets A and B has the same d_m-diameter as $R_{\alpha-1}$, i.e. equal to d. It is precisely this one of the two summands A and B that we take for R_α.

If, however, α is of the second kind, then we put $R_\alpha = \bigcap\limits_{\beta<\alpha} R_\beta$. Let us show that in this case as well, for the closed set R_α we have $d_m R_\alpha = d$.

Indeed, suppose the contrary. Let $d_m R_\alpha < d$. Take an infinite sequence of numbers $\alpha_1 < \alpha_2 < \cdots < \alpha_i < \cdots$ (i.e. a sequence of the type of the natural series of numbers), cofinal with a transfinite sequence of numbers β $(1 \leqslant \beta < \alpha)$. Then $R_\alpha = \bigcap_{i=1}^{\infty} R_{\alpha_i}$.

We have assumed that $d_m R_\alpha < d$. Therefore R_α can be covered by a finite number of closed subsets of diameter $< d$, forming a system of multiplicity $\leqslant m$ (i.e. no $m+1$ of these subsets have a common point). By slightly "inflating" these subsets (taking closures of spherical neighborhoods of sufficiently small radius of these subsets), we obtain closed sets of diameter still $< d$ and forming a system of multiplicity still $\leqslant m$. For sufficiently large $i = i_0$, the set $R_{\alpha_{i_0}}$ will be covered by these inflated elements and, consequently, $d_m R_{\alpha_{i_0}} < d$, which contradicts condition 3°.

Thus, the closed sets R_α are now defined for all $\alpha < \Omega$, and conditions 1°, 2°, and 3° are satisfied. Therefore there exists $\alpha_0 < \Omega$ such that

$$R_{\alpha_0} = R_{\alpha_0+1} = \cdots = R_\xi = \cdots \qquad (\alpha_0 < \xi < \Omega).$$

It is clear that R_{α_0} is a closed set which is no longer split by any closed subset of dimension $\leqslant k$ (and hence by no finite-dimensional subset, since in R, by hypothesis, there are no finite-dimensional subsets of dimension $> k$). And since $d_m R_{\alpha_0} = d > 0$, it follows that $\dim R_{\alpha_0} \geqslant m > k$. Therefore R_{α_0} is infinite-dimensional (all finite-dimensional subcompacta have, by hypothesis, dimension $\leqslant k$).

Thus R_{α_0} is an infinite-dimensional Cantor manifold lying in R, as was required to find.

Let us also note that, instead of transfinite induction, one could have used the well-known theorem of Brouwer on the existence, in a compactum possessing a certain inductive property, of a minimal compactum with the same property.

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