ON A CLASS OF INVERTIBLE OPERATORS IN THE RING OF ANALYTIC FUNCTIONS

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Abstract

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ON A CLASS OF INVERTIBLE OPERATORS IN THE RING OF ANALYTIC FUNCTIONS

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Let $K_m(r_i, R_i) = K_m$ be the ring of analytic functions of m complex variables $z_1, z_2, \dots, z_m$, regular and single-valued for $r_i < |z_i| < R_i$, $i = 1, 2, \dots, m$, in which a topology is defined by the notion of convergence as uniform convergence for $r_i(1+\varepsilon) < |z_i| < R_i(1-\varepsilon)$ for any $\varepsilon > 0$. In complete analogy with how this was done in ([1,2]), one can show that if K_m is considered only as a linear topological space, then the following assertion holds:

Theorem 1. Let A be a linear operator in K_m, defined by the equalities

$$Az_1^{n_1} \cdots z_m^{n_m} = z_1^{n_1} \cdots z_m^{n_m} \varepsilon_{n_1,\dots,n_m}(z_1, \dots, z_m), \qquad -\infty < n_1, \dots, n_m < \infty, \quad (1)$$

where $\varepsilon_{n_1,\dots,n_m}(z_1, \dots, z_m) \to 0$ as $\max_i |n_i| \to \infty$ (in the sense of the topology of K_m).

Then the operator $E + \lambda A$ has an inverse, continuous in K_m, for all λ except for a countable set of eigenvalues λ_n, having no limit points in the finite part of the plane. Moreover, the multiplicity of each eigenvalue is finite and, under a suitable ([2]) definition of the adjoint operator, all the Fredholm alternatives hold.

This result easily admits the following inessential generalization:

Theorem 2. Let the linear operator A_λ be defined by the equalities

$$A_\lambda z_1^{n_1} \cdots z_m^{n_m} = z_1^{n_1} \cdots z_m^{n_m} \varepsilon_{n_1,\dots,n_m}(z_1, \dots, z_m; \lambda), \qquad -\infty < n_1, \dots, n_m < \infty. \quad (2)$$

Here $\varepsilon_{n_1,\dots,n_m}(z_1, \dots, z_m; \lambda)$ are entire analytic functions of λ, satisfying the conditions

$$\lim_{\max_i |n_i| \to \infty} \varepsilon_{n_1,\dots,n_m}(z_1, \dots, z_m; \lambda) = 0 \quad (3)$$

uniformly in λ in any finite disk, and

$$\lim_{\lambda\to 0} \varepsilon_{n_1,\dots,n_m}(z_1,\dots,z_m;\lambda) = 0 \tag{4}$$

for fixed $n_1,\dots,n_m$.

Then the operator $E + A_\lambda$ has in K_m a continuous inverse for all λ, except for a countable set of eigenvalues λ_n, having no limit points in the finite part of the plane. Moreover, the finite multiplicity of the eigenvalues and the Fredholm alternatives still hold.

If K_m is considered not only as a linear topological space, but also as a topological ring, then one can obtain a considerably stronger result.

Theorem 3. Let B_λ be a linear operator in K_m, defined by the equalities

$$B_\lambda z_1^{n_1}\cdots z_m^{n_m} = z_1^{n_1}\cdots z_m^{n_m} h_{n_1,\dots,n_m}(z_1,\dots,z_m;\lambda), \qquad -\infty < n_1,\dots,n_m < \infty. \tag{5}$$

Here $h_{n_1,\dots,n_m}(z_1,\dots,z_m;\lambda)$ are entire analytic functions of λ, satisfying the conditions:

$$h_{n_1,\dots,n_m} \neq 0 \quad \text{for} \quad r_i(1+\varepsilon) < |z_i| < R_i(1-\varepsilon) \quad \text{for} \quad \max_i |n_i| > n_0(\varepsilon,\lambda), \tag{6}$$

$$\lim_{\max_i |n_i|\to\infty} \frac{h_{n_1+k_1,\dots,n_m+k_m}}{h_{n_1,\dots,n_m}} = 1 \tag{7}$$

uniformly with respect to λ in any finite circle for fixed $k_1,\dots,k_m$, and

$$\lim_{\lambda\to 0} h_{n_1,\dots,n_m} = 1 \tag{8}$$

for fixed $n_1,\dots,n_m$.

Then the operator B_λ has an inverse, continuous in K_m for all λ, except for a countable set λ_n, having no limit points in the finite part of the plane. The finite multiplicity of eigenvalues and the Fredholm alternatives still hold.

To compare the strength of Theorems 2 and 3, let us note that the operator $E + A_\lambda$, where A_λ is defined by the equalities (2), is easily represented in the form (5) by putting

$$h_{n_1,\dots,n_m}(z_1,\dots,z_m;\lambda) = 1 + \varepsilon_{n_1,\dots,n_m}(z_1,\dots,z_m;\lambda).$$

In this case, from condition (3) there follow not only conditions (6) and (7), but also the stronger assertion

$$\lim_{\max_i |n_i| \to \infty} h_{n_1,\ldots,n_m} = 1.$$

Theorem 3 essentially uses the fact that K_m is not only a linear topological space, but also a ring, and can be transferred (with these or other changes) to other topological or normed rings in which the powers of a finite number of elements form a basis under addition.

We do not give the proof because of lack of space. The main idea of the proof for the simplest case may be found in paper (3), a generalization of whose results is the present note.

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CITED LITERATURE

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2. M. A. Evgrafov, *Izv. AN SSSR, ser. matem.*, **21**, No. 2, 223 (1957).

3. M. A. Evgrafov, A. D. Solov' ev, *DAN*, **113**, No. 3 (1957).

Note: Figure translations are in progress. See original paper for figures.