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Abstract

Full Text

Mathematics

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ON THE QUESTION OF SOLVING THE LAPLACE EQUATION WITH AXIAL SYMMETRY BY THE DIFFERENCE METHOD

(Presented by Academician S. L. Sobolev on 18 XII 1956)

In note ([1]) we proposed a difference method for solving the axially symmetric Dirichlet problem for the Laplace equation

$$\Delta u = \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} = 0, \tag{1}$$

where r is the radial coordinate; z is the coordinate directed along the axis of symmetry. In order to obtain concrete difference equations, we constructed harmonic functions $\Phi_{2n-1}(r,z)$ $(n = 1, 2, \ldots)$ and $\Phi_{2n}(r,z)$ $(n = 0, 1, \ldots)$ with the aid of harmonic polynomials

$$P_k^*(r,z) = \left(\sqrt{r^2+z^2}\right)^k P_k\left(\frac{z}{\sqrt{r^2+z^2}}\right)$$

$$k = 0, 1, \ldots, \tag{2}$$

where $P_k(x)$ are Legendre polynomials. Below we obtain another form of the functions $\Phi_{2n-1}(r,z)$ and $\Phi_{2n}(r,z)$, and with the aid of these new functions we construct difference equations on 5 and on 9 points, which in this case turn out to be somewhat more accurate and more convenient for theoretical investigations.

1°. Let us set ourselves the goal of finding a function $u(r,z)$ which, in a domain G of the r,z-plane bounded by the contour Γ, satisfies equation (1), and on Γ takes prescribed values. With respect to the sought function $u(r,z)$, we shall assume, as in ([1]), that in the domain G it possesses continuous and bounded derivatives up to the order we need.

Cover the domain G by a square mesh with step h, and denote the coordinates of an arbitrary node by r_0, z_0. We shall continue to assume that the curve Γ intersects the straight lines forming the mesh exclusively at mesh nodes.

Lemma. If an arbitrary function $S(r,z)$ satisfies the relation

$$\Delta S(r,z) = -\frac{4}{r}\frac{\partial T(r,z)}{\partial r}, \tag{3}$$

where $T(r,z)$ is a harmonic function, then the function

$$Q(r,z) = S(r,z) + T(r,z)\ln r^2 \tag{4}$$

is harmonic.

Indeed, determining $\Delta Q(r,z)$, we have

$$\Delta Q(r,z) = \Delta S(r,z) + \ln r^2 \Delta T(r,z) + \frac{4}{r}\frac{\partial T(r,z)}{\partial r}.$$

Hence, taking into account (3) and the harmonicity of the function $T(r,z)$, we find $\Delta Q(r,z) = 0$, as was required to be proved.

In particular, if $T(r,z)$ is a homogeneous polynomial of degree k (2)

$$T(r,z) = P_k^*(r,z),$$

then, by virtue of (3), one may take as $S(r,z)$ also a homogeneous polynomial $S(r,z) = S_k(r,z)$ of the same degree k, and, consequently, in this case the function (4) will have the form

$$Q_k(r,z) = S_k(r,z) + P_k^*(r,z)\ln r^2. \tag{5}$$

We determine the coefficients of the polynomials $S_k(r,z)$ by the method of undetermined coefficients.

Using the harmonic polynomials (2) and the functions (5), one can construct functions $\Phi_{2n-1}(r,z)$ and $\Phi_{2n}(r,z)$, which for $n = 0,1,2,3,4$ have the form:

$$\Phi_0(r,z) = P_0^*, \qquad \Phi_1(r,z) = P_1^*,$$

$$\Phi_2(r,z) = \frac{r_0}{2}Q_0 - q_0 P_0^*, \qquad \Phi_3(r,z) = \frac{r_0}{2}Q_1 - q_0 P_1^*,$$

$$\Phi_4(r,z) = -\frac{1}{2}P_2^* - \frac{r_0^2}{4}Q_0 + q_1 P_0^*,$$

$$\Phi_5(r,z) = -\frac{1}{6}P_3^* - \frac{r_0^2}{4}Q_1 + q_1 P_1^*, \tag{6}$$

$$\Phi_6(r,z) = -\frac{r_0}{4}Q_2 + \frac{1}{2}q_0 P_2^* + \frac{r_0^3}{8}Q_0 + q_2 P_0^*,$$

$$\Phi_7(r,z) = -\frac{r_0}{12}Q_3 + \frac{1}{6}q_0 P_3^* + \frac{r_0^3}{8}Q_1 + q_2 P_1^*,$$

$$\Phi_8(r,z) = \frac{1}{24}P_4^* + \frac{r_0^2}{8}Q_2 - \frac{1}{2}q_1 P_2^* - \frac{r_0^4}{32}Q_0 - \frac{r_0^3}{16}\left(\frac{5}{4}r_0 - q_0\right)P_0^*,$$

where, for brevity of notation, we have denoted

$$P_k^* = P_k^*(r, z - z_0), \qquad Q_k = Q_k(r, z - z_0),$$

$$q_0 = -\frac{r_0}{2}\ln r_0^2, \qquad q_1 = \frac{r_0}{2}\left(q_0 - \frac{r_0}{2}\right), \qquad q_2 = \frac{r_0^2}{4}(r_0 - q_0),$$

and the functions $Q_k(r,z)$ $(k = 0,1,2,3)$, by virtue of (5), have the form:

$$Q_0(r,z) = \ln r^2, \qquad Q_1(r,z) = z\ln r^2,$$

$$Q_2(r,z) = r^2 + \left(z^2 - \frac{r^2}{2}\right)\ln r^2, \qquad Q_3(r,z) = 3zr^2 + \left(z^3 - \frac{3}{2}zr^2\right)\ln r^2.$$

Functions of the form (6) make it possible, by the method set forth in [1], to obtain difference equations for any number of involved nodes of an arbitrary grid.

For a square grid with step h, the difference relations composed from 5 points have the following form:

$$\begin{aligned} u(r_0,z_0) = b_1 u(r_0+h,z_0) + b_2 u(r_0-h,z_0) + \\ + b_3[u(r_0,z_0+h) + u(r_0,z_0-h)] + R, \end{aligned} \tag{7^1}$$

$$u(h,z_0) = \frac{2}{5}u(2h,z_0) + \frac{3}{10}[u(h,z_0+h) + u(h,z_0-h)] + R_1, \tag{7^2}$$

$$u(0,z_0) = \frac{2}{3}u(h,z_0) + \frac{1}{6}[u(0,z_0+h) + u(0,z_0-h)] + R_0, \tag{7^3}$$

where

$$b_1 = 4\xi^2\rho(\xi)\ln(1-\xi), \qquad b_2 = -4\xi^2\rho(\xi)\ln(1+\xi),$$

$$b_3 = \rho(\xi)E(\xi), \qquad \rho(\xi) = \frac{1}{2d(\xi)}, \qquad \xi = \frac{h}{r_0}, \qquad r_0 \geq 2h,$$

$$d(\xi) = E(\xi) + 2\xi^2 \ln\frac{1-\xi}{1+\xi}, \qquad E(\xi) = 2\xi\ln(1-\xi^2) + \xi^2\ln\frac{1-\xi}{1+\xi},$$

$$|R| \leq Ch^4M_4 + O(h^6), \qquad |R_1| \leq C_1h^4M_4 + O(h^6), \qquad |R_0| \leq C_0h^4M_4 + O(h^6).$$

Here C, C_1, and C_0 are completely definite constants, independent of the mesh step h and of u; M_4 is the upper bound of the moduli of the fourth derivatives of u in the open domain G, in which differentiation with respect to r is performed an even number of times.

The difference relations constructed from 9 points in the present case have the form

$$\begin{aligned} u(r_0, z_0) = b_1u(r_0+h, z_0) + b_2u(r_0-h, z_0) + b_3[u(r_0+h, z_0+h) + \\ + u(r_0+h, z_0-h)] + b_4[u(r_0, z_0+h) + u(r_0, z_0-h)] + \\ + b_5[u(r_0-h, z_0+h) + u(r_0-h, z_0-h)] + \overline{R}, \end{aligned} \tag{8^1}$$

$$\begin{aligned} u(h, z_0) = \frac{17}{53}u(2h, z_0) + \frac{7}{106}[u(2h, z_0+h) + u(2h, z_0-h)] + \\ + \frac{29}{106}[u(h, z_0+h) + u(h, z_0-h)] + R_1, \end{aligned} \tag{8^2}$$

$$\begin{aligned} u(0, z_0) = \frac{17}{29}u(h, z_0) + \frac{5}{58}[u(0, z_0+h) + u(0, z_0-h)] + \\ + \frac{7}{58}[u(h, z_0+h) + u(h, z_0-h)] + \overline{R}_0, \end{aligned} \tag{8^3}$$

where

$$b_1 = -2\rho(\xi)[S(\xi) - \xi^2\sigma_0(\xi)B(\xi)], \qquad b_2 = 2\rho(\xi)[T(\xi) - \xi^2\sigma_1(\xi)B(\xi)],$$

$$b_3 = \rho(\xi)S(\xi), \qquad b_4 = \rho(\xi)R(\xi), \qquad b_5 = -\rho(\xi)T(\xi),$$

$$\rho(\xi) = \frac{1}{2D(\xi)}, \qquad \xi = \frac{h}{r_0}, \qquad r_0 \geq 2h,$$

$$D(\xi) = R(\xi) + \xi^2[\sigma_0(\xi) - \sigma_1(\xi)]B(\xi),$$

$$S(\xi) = A(\xi)\gamma_1(\xi) + \sigma_0(\xi)\omega(\xi), \qquad T(\xi) = A(\xi)\gamma_2(\xi) + \sigma_1(\xi)\omega(\xi),$$

$$R(\xi) = \xi[B(\xi)\gamma_3(\xi) - 2A(\xi)] - [\sigma_0(\xi) - \sigma_1(\xi)][C(\xi) + A(\xi)],$$

$$A(\xi) = -B(\xi) + 4\sigma_0(\xi)\sigma_1(\xi), \qquad \omega(\xi) = C(\xi) + \frac{1}{6}\xi^2 B(\xi),$$

$$B(\xi) = (2+\xi)\sigma_0(\xi) + (2-\xi)\sigma_1(\xi),$$

$$C(\xi) = -4\sigma_0(\xi)\sigma_1(\xi) + \frac{1}{2}(3+\xi^2)[\sigma_0(\xi) + \sigma_1(\xi)] + \\ + \frac{1}{8}\xi(10+\xi^2)[\sigma_0(\xi) - \sigma_1(\xi)],$$

$$\gamma_1(\xi) = \sigma_0(\xi) + \xi\left(1 - \frac{1}{2}\xi\right), \qquad \gamma_2(\xi) = \sigma_1(\xi) - \xi\left(1 + \frac{1}{2}\xi\right),$$

$$\gamma_3(\xi) = \frac{1}{3}[B(\xi) + \sigma_0(\xi) + \sigma_1(\xi)], \qquad \sigma_0(\xi) = \ln(1-\xi), \qquad \sigma_1(\xi) = \ln(1+\xi),$$

$$|\overline{R}| \leq \overline{C}h^8 M_8 + O(h^{10}), \qquad |R_1| \leq C_1 h^8 M_8 + O(h^{10}),$$

$$|\overline{R}_0| \leq \overline{C}_0 h^6 M_6 + O(h^8).$$

Here $\overline{C}$, $\overline{C}_1$, and $\overline{C}_0$ are quite definite constants, independent of h and u; M_i $(i = 6, 8)$ is the upper bound of the moduli of the i-th derivatives of u in the open domain G for $r \geq h$ $(i = 8)$ and $r = 0$ $(i = 6)$, in which differentiation with respect to r is carried out an even number of times.

Discarding in equations (7) and (8) the remainder terms R, R_1, and R_0, and $\overline{R}$, $\overline{R}_1$, and $\overline{R}_0$, which represent quantities small in comparison with the remaining terms, we obtain two systems of equations in finite differences. Each of these systems makes it possible, for prescribed values of u on Γ, to determine approximate values of the function u at all interior nodes of the mesh of the domain G.

The system of linear equations may then be solved either by the iteration method or by the method of successive group elimination of the unknowns with the aid of matrix inversion.

We note that the difference equations obtained from (7^3) and (8^3) coincide with the corresponding equations given in ([2]) and ([1]), respectively.

It is also necessary to note that the coefficients b_i in formulas (7) and (8), for all $r_0 = nh$ ($n = 2, 3, \ldots$), are positive, whereas in the formula given in ([1]) some of them also take negative values.

2°. As an example, let us consider the problem of finding the electric field inside a cylindrical box of circular cross section of radius R, $0 \le z \le H$, both bases of which are grounded, while the lateral surface is charged to the potential V_0. Setting $H = R = V_0 = 1$ and choosing $h = 0.25$, we obtain, for the nodes located on the z-axis, the following results: by formulas (7), 0.17629808; 0.24598536; by formulas (8), 0.16464955; 0.2322553; by the formulas given in ([1]), 0.16471404; 0.23229634; by the formulas of S. A. Gershgorin ([2]), 0.17778664; 0.24792036; by the formula given by G. A. Grinberg in ([3]), 0.16453967; 0.23217446. Thus, the result obtained by formulas (7) is somewhat more accurate than the result obtained by the formulas of S. A. Gershgorin ([2]), while the result obtained by formulas (8) is more accurate than the result obtained by the formulas given in ([1]).

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